Chapter 3

Subgrid Scale Diffusion

Atmosphere is reperesented by grided point value in numerical simulation. However, air motions of smaller scale than that grid distance always exist in actual atmosphere. No matter how small we may get grid distance, such motions can exist. Those are called the subgrid scale motions, which act as diffusion in general. In addition, those correspond to turbulence and are often called 'turbulence diffusion'.

If the subgrid scale motion cannot be expressed when we use smaller grid distance, it is possible for us to obtain prognostic equations of the subgrid scale motion theoretically. For example, we can separate velocity into the corresponding means and deviation from that means. In the equations of averaged variables, the second-degree correlation of deviation appears as unknown variables which is called Reynolds stress. So we have to get their prognostic equations. However, the third-degree correlation appears in the prognostic equations. To repeat such operation produces many unknown variables, and a system of equations is not closed. Such problem is caused by unlinearity of turbulent flow. Kellar and Friedmann (1924) got first recognition on this problem, which is called 'closure problem'.

One of solution is that you rewrite odd unknown variables by known variables using equations limited. This solution is called 'closure assumptions'. Degree of predicted correlation decides what the solution is called, for example, one order closure, two order closure and so on. Furthermore, another solution is that you use only part of momentum equations system as closure assumptions. Thus, when it comes to usage of two-degree correlation, a expression of subgrid scale motion is made two separation.

- Modeling of unknown variables in the prognostic equations by dealing with two-degree correlation
- Modeling of the prognostic equations on the two-degree correlation represented by scalar, which can indicate averaged velocity, turbulence kinetic energy and turbulent flow of dissipation ratio through a concept of eddy viscosity

In this chapter, we discuss the formulation of subgrid scale motion with these two manner.

3.1 Parameteriaation of Turbulence Transport

There are various scale motion in atomosphere. The motion, which can be expressed with grid of numerical simulation, is called grid-scale motion, mean motion, subgrid scale motion or eddy motion.

To separate these motions, we assume that the field variables A, velocity, temperature, mixing ratio and so on, can be separated into mean field and deviation components.

$$A = \bar{A} + A'' \tag{3.1}$$

where the corresponding means are indicated by and the deviation components by $''$.

There are various ways when we average variables, but we do not show them in detail. For your information, average of deviations and average of products of two variabeles are given by

$$\overline{A''} = 0 \tag{3.2}$$
$$\overline{AB} = \bar{A}\bar{B} + \overline{A''B''} \tag{3.3}$$

In other words, if you average products, you do not always get the variable which correspond with products of each averaged, and the second term of above equation appears. We apply them to x components in equations of motion. Here, to simplify we discuss incompressible fluid $\rho = const$, and we can obtain a variable consisting of mean and deviation components.

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} - f\bar{v} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} \\ -\frac{1}{\rho}\left(\frac{\partial}{\partial x}\rho\overline{u''u''} + \frac{\partial}{\partial y}\rho\overline{u''v''} + \frac{\partial}{\partial z}\rho\overline{u''w''}\right) + \nu\nabla^2\bar{u}\end{aligned} \tag{3.4}$$

The terms $-\rho\overline{u''u''}, -\rho\overline{u''v''}, -\rho\overline{u''w''}$ in this equation represent stress by turbulent flow, which is called eddy stress or Reynolds stress. We can regard them as transport of momentum, so stress are made by trasport of momentum by eddy.

Similarly, the equations on potential temperature and mixing ratio are given by

$$\frac{\partial \bar{\theta}}{\partial t} + \bar{u}\frac{\partial \bar{\theta}}{\partial x} + \bar{v}\frac{\partial \bar{\theta}}{\partial y} + \bar{w}\frac{\partial \bar{\theta}}{\partial z} = -\frac{\partial}{\partial x}\overline{u''\theta''} - \frac{\partial}{\partial y}\overline{v''\theta''} - \frac{\partial}{\partial z}\overline{w''\theta''} \tag{3.5}$$

Even if you use grid, the variables indicated by prime cannot be expressed. If you use them in numerical simulation, you have to represent them with the variables indicated by overbar (mean components). If you not, you cannot get effects of transport by eddy motion which has smaller scale than grid distance. The problem of turbulent parametarization is how we have to express such deviation by eddy, which is indicated by prime, by using limited grid. In addition, you need to pay attention to a difference between the definition of these variables indicated by and the definition of variables indicated by shown at the second chapter.

3.2 Eddy Viscosity Model

3.2.1 Formulation of Diffusion Term

In this section, we formulate diffusion term (the term of turbulence mixing) $G^{\frac{1}{2}}\mathrm{Turb}.\phi$. The diffusion term appears in Equations of motion (??)(??), Equation of potential temperature (??), Equations of mixing ratio of water vapor and hydrometeor (??) and Equations of number concentration per unit volume (??), which shown in the basic dynamical equations in terrain-following coordinates at the section ??.

The diffusion term in the equation of motions is expressed using stress tensor τ_{ij} as follows:

$$\begin{aligned} G^{\frac{1}{2}}\mathrm{Turb}.u &= G^{\frac{1}{2}}\left(\frac{\partial\tau_{11}}{\partial x}+\frac{\partial\tau_{12}}{\partial y}+\frac{\partial\tau_{13}}{\partial z}\right) \\ &= \frac{\partial}{\partial\xi}\left(J_d\tau_{11}\right)+\frac{\partial}{\partial\eta}\left(J_d\tau_{12}\right)+\frac{\partial}{\partial\zeta}\left(\tau_{13}+J_{31}\tau_{11}+J_{32}\tau_{12}\right) \end{aligned} \tag{3.6}$$

$$\begin{aligned} G^{\frac{1}{2}}\mathrm{Turb}.v &= G^{\frac{1}{2}}\left(\frac{\partial\tau_{21}}{\partial x}+\frac{\partial\tau_{22}}{\partial y}+\frac{\partial\tau_{23}}{\partial z}\right) \\ &= \frac{\partial}{\partial\xi}\left(J_d\tau_{21}\right)+\frac{\partial}{\partial\eta}\left(J_d\tau_{22}\right)+\frac{\partial}{\partial\zeta}\left(\tau_{23}+J_{31}\tau_{21}+J_{32}\tau_{22}\right) \end{aligned} \tag{3.7}$$

$$\begin{aligned} G^{\frac{1}{2}}\mathrm{Turb}.w &= G^{\frac{1}{2}}\left(\frac{\partial\tau_{31}}{\partial x}+\frac{\partial\tau_{32}}{\partial y}+\frac{\partial\tau_{33}}{\partial z}\right) \\ &= \frac{\partial}{\partial\xi}\left(J_d\tau_{31}\right)+\frac{\partial}{\partial\eta}\left(J_d\tau_{32}\right)+\frac{\partial}{\partial\zeta}\left(\tau_{33}+J_{31}\tau_{31}+J_{32}\tau_{32}\right) \end{aligned} \tag{3.8}$$

where stress tensor τ_{ij} consists of shear stress and Reynolds stress. Reynolds stress consists of fluctuation from averaged variables, so we need modeling in any way to the form with averaged variables. Reynolds stress can express in the form of gradient diffusion using viscosity coefficient from an analogy of shear stress.

$$\tau_{11} = \bar{\rho}\nu_{\tau h}\left(S_{11}-\frac{2}{3}Div\right) \tag{3.9}$$

$$\tau_{12} = \bar{\rho}\nu_{\tau h}S_{12} \tag{3.10}$$

$$\tau_{13} = \bar{\rho}\nu_{\tau v}S_{13} \tag{3.11}$$

$$\tau_{21} = \bar{\rho}\nu_{\tau h}S_{12} \tag{3.12}$$

$$\tau_{22} = \bar{\rho}\nu_{\tau h}\left(S_{22}-\frac{2}{3}Div\right) \tag{3.13}$$

$$\tau_{23} = \bar{\rho}\nu_{\tau v}S_{23} \tag{3.14}$$

$$\tau_{31} = \bar{\rho}\nu_{\tau h}S_{13} \tag{3.15}$$

$$\tau_{32} = \bar{\rho}\nu_{\tau h}S_{23} \tag{3.16}$$

$$\tau_{33} = \bar{\rho}\nu_{\tau v}\left(S_{33}-\frac{2}{3}Div\right) \tag{3.17}$$

where $\nu_{\tau h} and \nu_{\tau v}$ are horizontal and vertical eddy viscosity coefficients regarding kinetic momentum, respectively. The molecular viscosity coefficient of shear stress is so small to eddy viscosity coefficient that it can be neglected. S_{ij} is deformation rate tensor. In curvilinear coordinate system, it is given by

$$S_{11} = 2\frac{\partial u}{\partial x} = \frac{2}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \xi}(J_d u) + \frac{\partial}{\partial \zeta}(J_{31} u)\right] \tag{3.18}$$

$$S_{22} = 2\frac{\partial v}{\partial y} = \frac{2}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \eta}(J_d v) + \frac{\partial}{\partial \zeta}(J_{32} v)\right] \tag{3.19}$$

$$S_{33} = 2\frac{\partial w}{\partial z} = \frac{2}{G^{\frac{1}{2}}}\frac{\partial w}{\partial \zeta} \tag{3.20}$$

$$S_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \eta}(J_d u) + \frac{\partial}{\partial \xi}(J_d v) + \frac{\partial}{\partial \zeta}(J_{32} u + J_{31} v)\right] \tag{3.21}$$

$$S_{13} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \xi}(J_d w) + \frac{\partial}{\partial \zeta}(u + J_{31} w)\right] \tag{3.22}$$

$$S_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \eta}(J_d w) + \frac{\partial}{\partial \zeta}(v + J_{32} w)\right] \tag{3.23}$$

and Div is divergence.

$$Div = \frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \xi}\left(G^{\frac{1}{2}} u\right) + \frac{\partial}{\partial \eta}\left(G^{\frac{1}{2}} v\right) + \frac{\partial}{\partial \zeta}\left(G^{\frac{1}{2}} W\right)\right] \tag{3.24}$$

The diffusion terms of potential temperature, mixing ration of hydrometeor and water vapor and the number concentration per unit volume are formalized by using ϕ as follows:

$$\begin{aligned} G^{\frac{1}{2}}\mathrm{Turb}.\phi &= G^{\frac{1}{2}}\left(\frac{\partial H_{\phi 1}}{\partial x} + \frac{\partial H_{\phi 2}}{\partial y} + \frac{\partial H_{\phi 3}}{\partial z}\right) \\ &= \frac{\partial}{\partial \xi}(J_d H_{\phi 1}) + \frac{\partial}{\partial \eta}(J_d H_{\phi 2}) + \frac{\partial}{\partial \zeta}(H_{\phi 3} + J_{31} H_{\phi 1} + J_{32} H_{\phi 2}) \end{aligned} \tag{3.25}$$

where $H_{\phi 1}, H_{\phi 2}$ and $H_{\phi 3}$ are the molecular diffusions of the corresponding scalar ϕ and turbulent fluxes in the x, y and z directions, respectively. They have forms of the gradient diffusion similar to velocity

$$H_{\phi 1} = \bar{\rho}\nu_{Hh}\frac{\partial \phi}{\partial x} = \bar{\rho}\nu_{Hh}\frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \xi}(J_d \phi) + \frac{\partial}{\partial \zeta}(J_{31}\phi)\right] \tag{3.26}$$

$$H_{\phi 2} = \bar{\rho}\nu_{Hh}\frac{\partial \phi}{\partial y} = \bar{\rho}\nu_{Hh}\frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \eta}(J_d \phi) + \frac{\partial}{\partial \zeta}(J_{32}\phi)\right] \tag{3.27}$$

$$H_{\phi 3} = \bar{\rho}\nu_{Hv}\frac{\partial \phi}{\partial z} = \bar{\rho}\nu_{Hv}\frac{1}{G^{\frac{1}{2}}}\frac{\partial \phi}{\partial \zeta} \tag{3.28}$$

where $\nu_{Hh} and \nu_{Hv}$ are horizontal and vertical eddy viscosity coefficients about scalar, respectively. The molecular diffusion coefficient is so small that it is neglected.

We have the modeling of Reynolds to the form with eddy viscosity coefficient. The manner is called eddy viscosity model, with which we can appreciate the eddy viscosity coefficients $\nu_{\tau h} and \nu_{\tau v} and the essy diffusion coefficients \nu_{Hh} an$ appeard in avobe equations.

In following section, two eddy viscosity model are explained, which are actually used in ***CReSS***.

- One order closure of Smagorinsky
- One and a half order closure with turbulence kinetic energy

3.2.2 One order closure of Smagorinsky

Smagorinsky (1963) and Lilly (1962) give the eddy viscosity coefficient in the case it is isotropic vertically and horizontally, using $\nu_{\tau h} = \nu_{\tau v} = \nu_\tau$

$$\nu_\tau = \begin{cases} (C_S\Delta)^2 \left(Def^2 - \dfrac{N^2}{Pr}\right), & \nu_\tau > 0 \\ 0, & \nu_\tau \le 0 \end{cases} \tag{3.29}$$

where C_S is smagorinsky constant, $C_S = 0.21$ by Deardorff (1972a). Δ is averaged grid interval of numerical simulation.

$$\Delta = (\Delta x \Delta y \Delta z)^{\frac{1}{3}} \tag{3.30}$$

Def, which is measurement of transformation, can be obtained

$$Def^2 = \frac{1}{2}\left(S_{11}^2 + S_{22}^2 + S_{33}^2\right) + S_{12}^2 + S_{13}^2 + S_{23}^2 - \frac{2}{3}Div^2 \tag{3.31}$$

and

$$N^2 = \begin{cases} \dfrac{g}{G^{\frac{1}{2}}} \dfrac{\partial \ln\theta}{\partial \zeta}, & q_v < q_{vsw} \\ \dfrac{g}{G^{\frac{1}{2}}} \left[\dfrac{1 + \dfrac{\mathcal{L}_v q_{vsw}}{R_d T}}{1 + \dfrac{\mathcal{L}_v^2 q_{vsw}}{C_p R_v T^2}} \left(\dfrac{\partial \ln\theta}{\partial \zeta} + \dfrac{\mathcal{L}_v}{C_p T} \dfrac{\partial q_{vsw}}{\partial \zeta} \right) - \dfrac{\partial q_w}{\partial \zeta} \right], & q_v \ge q_{vsw} \end{cases} \tag{3.32}$$

where N is a constant Brunt-Väisälä frequency, and

$$Pr = \frac{\nu_\tau}{\nu_H}, \qquad \nu_{Hh} = \nu_{Hv} = \nu_H \tag{3.33}$$

where Pr is Turbulent Prandtl number. So we can obtain the eddy viscosity coefficient regarding scalar ϕ. g is gravity acceleration, T is temperature, $R_d and R_v$ are gas constants for dry air and wet air, respectively. C_p is specific heat at constant pressure and q_w is a sum of mixing ratio of molecular weight for water vapor, cloud liquid water and rainwater. With the equation of Tetens, mixing ratio of water saturation q_{vsw} is given by

$$q_{vsw} = \epsilon \frac{610.78}{p} \exp\left(17.269 \frac{T - 273.16}{T - 35.86}\right) \tag{3.34}$$

and latent heat for water evaporation $\mathcal{L}_v$ is given by

$$\mathcal{L}_v = 2.50078 \times 10^6 \left(\frac{273.16}{T}\right)^{\left(0.167 + 3.67 \times 10^{-4} T\right)} \tag{3.35}$$

where ϵ is the ratio of molecular weight of water vapor and of dry air.

Next, in the case it is anisotropic vertically and horizontally, the eddy viscosity coefficients of each direction are represented as

$$\nu_{\tau h} = \begin{cases} (C_S \Delta_h)^2 \left(Def^2 - \dfrac{N^2}{Pr}\right), & \nu_{\tau h} > 0 \\ 0, & \nu_{\tau h} \leq 0 \end{cases} \tag{3.36}$$

$$\nu_{\tau v} = \begin{cases} (C_S \Delta_v)^2 \left(Def^2 - \dfrac{N^2}{Pr}\right), & \nu_{\tau v} > 0 \\ 0, & \nu_{\tau v} \leq 0 \end{cases} \tag{3.37}$$

where Δ_h and Δ_v are given by

$$\Delta_h = (\Delta x \Delta y)^{\frac{1}{2}} \tag{3.38}$$

$$\Delta_v = \Delta z \tag{3.39}$$

Furthermore,

$$Pr = \frac{\nu_{\tau h}}{\nu_{Hh}} = \frac{\nu_{\tau v}}{\nu_{Hv}} \tag{3.40}$$

Thus, we obtain the eddy viscosity coefficients ν_{Hh}, ν_{Hv} of each direction regarding scalar ϕ, using the Turbulent Prandtl number Pr.

3.2.3 One and a half order closure with turbulence kinetic energy

With regard to one and a half order closure, we use prognostic equations on turbulence kinetic energy. We mark deviation from averaged flow each velosity component with $''$, so this turbulence kinetic energy is represented as

$$E = \frac{1}{2}\left(\overline{u''^2 + v''^2 + w''^2}\right) \tag{3.41}$$

and, the prognostic equations are given by

$$\begin{aligned}\frac{\partial \rho^* E}{\partial t} &= -\left(u^*\frac{\partial E}{\partial \xi} + v^*\frac{\partial E}{\partial \eta} + W^*\frac{\partial E}{\partial \zeta}\right) + \text{Buoy}.E + \rho^*\left(\frac{1}{2}\nu_E Def^2 - \frac{2}{3}EDiv\right) - \rho^*\frac{C_e}{l_h}E^{\frac{3}{2}} \\ &+ \left[\frac{\partial}{\partial \xi}(J_d H_{E1}) + \frac{\partial}{\partial \eta}(J_d H_{E2}) + \frac{\partial}{\partial \zeta}(H_{E3} + J_{31}H_{E1} + J_{32}H_{E2})\right]\end{aligned} \tag{3.42}$$

Buoy.E which appear in equations (3.42), is given as follows with the converting term of potential energy and kinetic energy.

$$\text{Buoy}.E = \begin{cases} -g\rho^*\nu_{Hv}\left(A\dfrac{\partial \theta_e}{\partial \zeta} - \dfrac{\partial q_{all}}{\partial \zeta}\right), & q_v \geq q_{vsw} \;\; or \;\; q_c + q_i > 0 \\ -g\rho^*\nu_{Hv}\left(\dfrac{1}{\theta}\dfrac{\partial \theta}{\partial \zeta} + 0.61\dfrac{\partial q_v}{\partial \zeta}\right), & q_v < q_{vsw} \;\; or \;\; q_c + q_i = 0 \end{cases} \tag{3.43}$$

where, A is expressed as

$$A = \frac{1}{\theta}\frac{1 + \dfrac{1.61\epsilon\mathcal{L}_v q_v}{R_d T}}{1 + \dfrac{\epsilon\mathcal{L}_v^2 q_v}{C_p R_d T^2}} \tag{3.44}$$

where, g is gravity acceleration, T is temperature, ϵ is the ratio of molecular weight for water vapor and dry air, q_{all} is a sum of mixing ratio of molecular weight for water vapor, cloud liquid water and cloud ice, and θ_e is equivalent potential temperature. $C_p and R_d$ are gas constants for dry air and wet air, respectively. $\mathcal{L}_v$ is latent heat for water evaporation.

Next, the third term of right hand Def, Div is shown at the section 3.2.2. The coefficient C_e of the forth term, the dissipating term is represented as

$$C_e = \begin{cases} 3.9, & thelowestlayer \\ 0.93, & theotherlayer \end{cases} \tag{3.45}$$

In addition, the last term of right hand, flux of turbulence kinetic energy is given by

$$H_{E1} = \bar{\rho}\nu_E\frac{\partial E}{\partial x} = \bar{\rho}\nu_E\frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \xi}(J_d E) + \frac{\partial}{\partial \zeta}(J_{31}E)\right] \tag{3.46}$$

$$H_{E2} = \bar{\rho}\nu_E\frac{\partial E}{\partial y} = \bar{\rho}\nu_E\frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \eta}(J_d E) + \frac{\partial}{\partial \zeta}(J_{32}E)\right] \tag{3.47}$$

$$H_{E3} = \bar{\rho}\nu_E\frac{\partial E}{\partial z} = \bar{\rho}\nu_E\frac{1}{G^{\frac{1}{2}}}\frac{\partial E}{\partial \zeta} \tag{3.48}$$

where ν_E is the eddy viscosity coefficient for turbulence kinetic energy.

Thus, the eddy viscosity coefficients $\nu_{\tau h}, \nu_{\tau v}$ are represented as a function of turbulent kinetic energy E,

$$\nu_{\tau h} = 0.1E^{\frac{1}{2}} l_h \quad (3.49)$$
$$\nu_{\tau v} = 0.1E^{\frac{1}{2}} l_v \quad (3.50)$$

where $l_h and l_v$ are horisontal and vertical mixing length scales. In the case where the difference is small between horisontal and vertical grid interval,

$$l = l_h = l_v = \begin{cases} \Delta s, & unstable or neutral \\ min\left(\Delta s, l_s\right), & stable \end{cases} \quad (3.51)$$

where $Deltas and l_s are expressed as

$$\Delta s = \Delta s_h = \Delta s_v = (\Delta x \Delta y \Delta z)^{\frac{1}{3}} \quad (3.52)$$
$$l_s = 0.76E^{\frac{1}{2}} \left| \frac{g}{\bar{\theta}} \frac{\partial \bar{\theta}}{\partial z} \right|^{-\frac{1}{2}} \quad (3.53)$$

On the other hand, in the case where the difference is large between horisontal and vertical grid interval,

$$l_h = \Delta s_h \quad (3.54)$$

$$l_v = \begin{cases} \Delta s_v, & unstable or neutral \\ min\left(\Delta s_v, l_s\right), & stable \end{cases} \quad (3.55)$$

where Δs_h and Δs_v are expressed as

$$\Delta s_h = (\Delta x \Delta y)^{\frac{1}{2}} \quad (3.56)$$
$$\Delta s_v = \Delta z \quad (3.57)$$

Finally, the eddy viscosity coefficients $\nu_{\tau h} and \nu_{\tau v}$ are represented as a function of turbulence kinetic energy E,

$$\nu_{\tau h} = max\left(0.1E^{\frac{1}{2}} l_h, \alpha \Delta s_h^2\right) \quad (3.58)$$
$$\nu_{\tau v} = max\left(0.1E^{\frac{1}{2}} l_v, \alpha \Delta s_v^2\right) \quad (3.59)$$

where α is smaller number like $\alpha = 10^{-6}$. The eddy viscosity coefficient ν_{Hv}, ν_{Hh} for scalar ϕ and ν_E for the turbulence kinetic energy E can be obtained as follows:

In the case of the small difference between horisontal and vertical grid interval,

$$\frac{\nu_{\tau h}}{\nu_{Hh}} = \frac{\nu_{\tau v}}{\nu_{Hv}} = \frac{1}{1 + 2l/\,\Delta s} \;\; (\;= Pr \;) \tag{3.60}$$

$$\nu_E = 2\nu_{\tau h} = 2\nu_{\tau v} \tag{3.61}$$

In the case of the large difference between horisontal and vertical grid interval,

$$\frac{\nu_{\tau h}}{\nu_{Hh}} = \frac{1}{1 + 2l_h/\,\Delta s_h} = \frac{1}{3} \tag{3.62}$$

$$\frac{\nu_{\tau v}}{\nu_{Hv}} = \frac{1}{1 + 2l_v/\,\Delta s_v} \;\; (\;= Pr \;) \tag{3.63}$$

$$\nu_E = 2\nu_{\tau h} \tag{3.64}$$