Chapter 6

Numerical computation method

Generally, the equation of motion, the equation of potential temperature derived by the equation of thrmodynamics, the equation of pressure derived by the continity eqution and the eqution of gaseous state, and the equations of water vapor and hydrometeors are solved simultaneously and integrated with respect to time under appropriate boundary and initial values in cloud resolving models.

In **CReSS**, the equation of motion, the equations of the potential temperature perturbation and the pressure perturbation, and the equations of hydrometeors and water vapor which are shown in Chapter 2 'Formulation of the system of basic equations' are also temporally and spatially integrated by using finite difference method. There are many studies about the methods which are discretizated and integrated these equation systems and the various calculation techniques are established.

In this chapter, the structure of a grid point, the method of discretization, the method of time integration and so on are explained about the numerical computation method used by **CReSS**.

6.1   The discretization of basic equation system

6.1.1   The outline of the numerical solution

There are various techniques from the selection of dependent variables to the method of time integration in the numerical solution by cloud resolving models. The outline of the numerical solution used in **CReSS** is as follows.

- The basic equation system is the equation of motion, potential temperature perturbation, pressure perturbaton, water vapor and hydrometeors.

- The dependent variables are three-dimensionl velocity components u, v, w, potential temperature perturbation θ', pressure perturbation p', mixing ratio of watar vapor, and mixing ratios and number concentrations of hydrometeors.

- The basic equations are defined by a terrain-following curvilinear coordinates are used to include the effect of orography.

- The finite differences are used for the derivation of the dependent variable with respect to space. The types of Arakawa-C and Lorenz staggered grids are used for hrizontal and vertical grid arrangement, respectively.

- The sound wave is contained as a solution by using compressible equation system. The terms related to the sound wave are integrated with small time step $\Delta\tau$, the other terms are integrated with large time step Δt.

- The leap-frog scheme with the Asselin time filter is used for time integration with respect to the large time step.

- The explicited forward-backward scheme or the inplicited Crank-Nicolson scheme in only vertical is used for time integration with respect to the small time step.

- The effects of sound wave are weakened by adding the divergence terms explicit scheme with the small time step.

Furthermore, ξ, η are expressed as x, y in this chapter as it is $\xi = x, \eta = y$ in a terrain-following coordinates.

6.1.2   Grid in the model and setting of variables

As the space are represented by the method of grid points in **CReSS**, finite difference method is used for the spatial derivation. Here, the strucuture of grid points and the setting of variables in **CReSS** are shown.

The strucuture of grid points and the setting of variables are set on the staggered grid [1] as Fig6.1 in horizontal and vertical. Arakawa-C grid and Lorenz grid are used for the settings of horizontal and vertical grid points, respectively.

In this case, the point of the all scalars, such as pressure perturbations, potential temperature perturbation, mixing ratios of watar vapor and physical quantities of hydrometeors, are located at the center of the grid box[2].

The point of x-component velocity u is dislocated to a half interval from the center of the grid in the x-direction, the point of y-component velocity v is dislocated to a half interval from the center of the grid in the y-direction in horizontal. The point of u and v are defined at the same level as the scalars points. The point of z-component velocity w is vertically dislocated at a half interval just above and below the scalar variables points.

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[1]There are various setting in the staggered grid which variables are defined alternately.

[2]The point of potential temperature (temperature) and the point of pressure are located at the same level in Lorenz grid. On the other hand, the point of potential temperature (temperature) is dislocated at a half interval below the pressure point, and defined at the same level as the vertical velocity point in the Charney Phillips' grid.
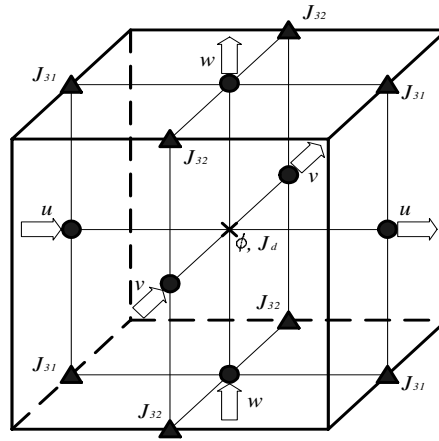
Figure 6.1. Structure of the staggered grid and setting of dependent variables.

Furthermore, as a terrain-following coordinate is used in **CReSS**, the Jacobian with transform for coordinates $G^{\frac{1}{2}}$ and the metric tensor which is not zero or 1, that is, J_{31}, J_{32}, J_d $(= G^{\frac{1}{2}})$ are shown as follows. The $G^{\frac{1}{2}}$ $(= J_d)$ point is defined at the same level as the scalars points. The J_{31} point is dislocated to a half interval from the center of the grid in the x-direction and the z-direction. The J_{32} is dislocated to a half interval from the center of the grid in the y-direction and the z-direction.

The coordinates x, y, ζ which are set on the staggered grid as shown in Fig 6.1 are difined as the vectors points. The domain in a thick line except the most outside grid points is the physical domain as shown in Fig6.2. So four grid points in the direction without a dimension need to be calculated at the time of the 2-dimensional caluculation.

6.1.3   Discretization by finite difference method

Generally, since elastic equations without any approximations are used in the cloud resolving model as the basic equations, all waves in the atmosphere are represented in the model. Waves in the atmosphere are divided into the sound wave, the gravity wave and the Rossby wave by phase velocity or frequency, and the phase velocities become small in this order. The phase velocity of sound wave is the fastest but we have no meteorological interest for it. On the other hand, the gravity wave and the Rossby wave are important for a cloud and a mesoscale meteorological phenomena[3] However, the sound wave is contained in models with elastic equation system, and the special technique is needed to be used for the calculation about the wave. So the time step is very small to satisfy CFL condition for the sound wave which the phase velocity is fastest if the time integration is calculated explicitly in meteorological models the generall numerical integration for fluid. However, the time steps for the gravity and the Rossby waves which are meteorologically important are satisfied with the much larger time step than that of the sound wave, since the phase velocities of these waves are much slower than the sound wave.

There are various calculation techniques not to make the time step of the sound wave small[4]. The well-known calculation techniques are as follows.

HI-VI method   The terms of the sound wave are solved inplicitly, and the other terms are solved explicitly. This technique is called semi-inplicit time integration method[5].

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[3]The Rossby wave is important for larger scale phenomena in mesocale meteorology, synoptic meteorological phenomena and planetary meteorological phenomena. Weather is influenced by the Rossby-scale atmospheric phenomena every day, and usual forecast models represent these phenomena.

[4]If the anelastic equation system by Ogura and Phillips (1962) is used for the basic equation system in a numerical model, the time step can be defined by the phase velocity of gravity wave because the sound wave is not contained in this equation system. In this case, pressure is calculated by the equation of motion and the continity eqution. There are the advantage that the sound wave is removed, but the demerits of which the equation of pressure is complicated and a error is large at the place with large variations of density in this techniques.

[5]The terms with large time step are explicitly calculated and the other terms are inplicitly solved in the semi-inplicit

HE-VE method	The terms of the sound wave and the other terms are solved explicitly using different time steps, respectively. This technique which is called the mode-splitting technique (Klemp and Wilhelmson, 1978) is often used.
HE-VI method	This technique is almost the same as the mode-splitting technique. But the terms of the sound wave are solved inplicitly in vertical.

The schematic representation of this mode-splitting technique is shown in Fig6.3.

Many data are exchanged between some nodes in the case of using parallel computer since large simultaneous equations are solved in the HI-VI method. The HE-VE method or HE-VI method are used in **CReSS** since it is assumed to be mainly used on a parallel computer. These techniques are easy to be applied to the large parallel computing since the phisycal space are corrsponded to the setting of a node and only the data in halo region of a node communicates between nodes. The detail description is shown in 7 'Mounting of a parallel processing'. The efficiency of calculation does not become so bad to integrate the terms with respect to sound wave in small time step.
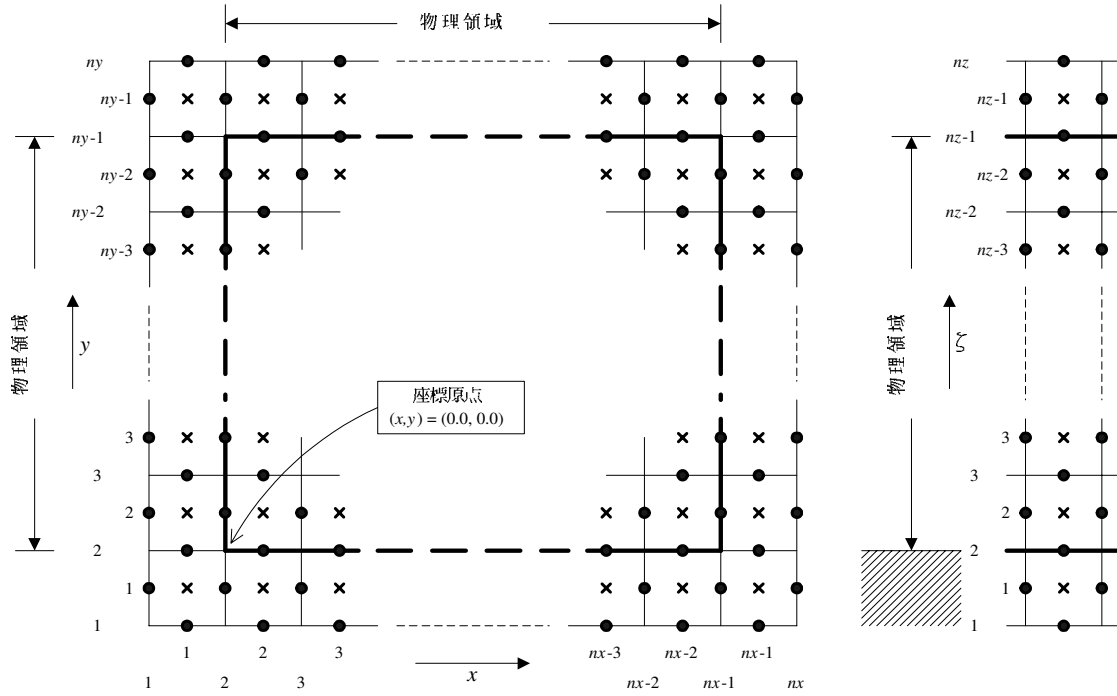
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time integration method. For example, the semi-inplicit time integration method of which the terms for the gravity wave are explicitly solved, and the other terms are solved by using the leap-frog scheme is used in the model of JMA.

Figure 6.2. Setting of variables in a calculation domain (● represents a point for velocities. ✕ represents a point for scalars).
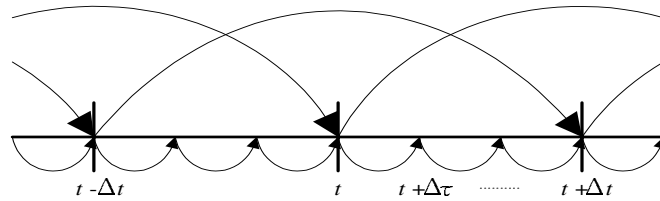

Figure 6.3. The schematic representation of the mode-splitting technique.

The method of discretization for physical quanities with the sound wave mode

The method of discretization for physical quantities with the sound wave mode is shown from here. The gravity wave mode is calculated with the large time step. In this case, the discretizations for potential temperature, mixing ratios of watar vapor, mixing ratios and number concentrations of hydrometeors is not took into consideration as they are calculated in the large time step.

Physical quantities with the sound wave mode are shown in the basic equation system in **CReSS**. In the following equations, $[rm]$ is indicates terms which are related to the Rossby wave mode (rotational mode), $[gm]$ the gravity wave mode (divergence mode) and $[am]$ the sound wave mode, respectively. Terms of physical processes are indicated by $[physics]$.

The equation of motion

$$\frac{\partial u^*}{\partial t} = \underbrace{-\left(u^* \frac{\partial u}{\partial x} + v^* \frac{\partial u}{\partial y} + W^* \frac{\partial u}{\partial \zeta} \right)}_{[rm]}$$

$$\underbrace{-\left[\frac{\partial}{\partial x} \left\{ J_d \left(p' - \alpha Div^* \right) \right\} + \frac{\partial}{\partial \zeta} \left\{ J_{31} \left(p' - \alpha Div^* \right) \right\} \right]}_{[am]} + \underbrace{(f_s v^* - f_c w^*)}_{[rm]} + \underbrace{G^{\frac{1}{2}} \mathrm{Turb}.u}_{[physics]} \qquad (6.1)$$

$$\frac{\partial v^*}{\partial t} = \underbrace{-\left(u^* \frac{\partial v}{\partial x} + v^* \frac{\partial v}{\partial y} + W^* \frac{\partial v}{\partial \zeta} \right)}_{[rm]}$$

$$\underbrace{-\left[\frac{\partial}{\partial y} \left\{ J_d \left(p' - \alpha Div^* \right) \right\} + \frac{\partial}{\partial \zeta} \left\{ J_{32} \left(p' - \alpha Div^* \right) \right\} \right]}_{[am]} - \underbrace{f_s u^*}_{[rm]} + \underbrace{G^{\frac{1}{2}} \mathrm{Turb}.v}_{[physics]} \qquad (6.2)$$

$$\frac{\partial w^*}{\partial t} = \underbrace{-\left(u^* \frac{\partial w}{\partial x} + v^* \frac{\partial w}{\partial y} + W^* \frac{\partial w}{\partial \zeta} \right)}_{[rm]}$$

$$\underbrace{-\frac{\partial}{\partial \zeta} \left(p' - \alpha Div^* \right)}_{[am]} - \rho^* g \left(\underbrace{\frac{\theta'}{\bar{\theta}}}_{[gm]} - \underbrace{\frac{p'}{\bar{\rho} c_s^2}}_{[am]} + \underbrace{\frac{q_v'}{\epsilon + \bar{q}_v} - \frac{q_v' + \sum q_x}{1 + \bar{q}_v}}_{[physics]} \right) + \underbrace{f_c u^*}_{[rm]} + \underbrace{G^{\frac{1}{2}} \mathrm{Turb}.w}_{[physics]} \qquad (6.3)$$

The equation of pressure

$$\frac{\partial G^{\frac{1}{2}} p'}{\partial t} = \underbrace{-\left(G^{\frac{1}{2}} u \frac{\partial p'}{\partial x} + G^{\frac{1}{2}} v \frac{\partial p'}{\partial y} + G^{\frac{1}{2}} W \frac{\partial p'}{\partial \zeta} \right)}_{[rm]} + \underbrace{G^{\frac{1}{2}} \bar{\rho} g w}_{[am]}$$

$$\underbrace{- \bar{\rho} c_s^2 \left(\frac{\partial G^{\frac{1}{2}} u}{\partial x} + \frac{\partial G^{\frac{1}{2}} v}{\partial y} + \frac{\partial G^{\frac{1}{2}} W}{\partial \zeta} \right)}_{[am]} + \underbrace{G^{\frac{1}{2}} \bar{\rho} c_s^2 \left(\frac{1}{\theta} \frac{d\theta}{dt} - \frac{1}{Q} \frac{dQ}{dt} \right)}_{[am]} \qquad (6.4)$$

These equation systems are made discrete by using finite difference method and integrated for the grid system as Fig 6.1. However, the equalization is required to calculate at the same point as using the

staggered grid. So the equalization operator is defined as follows. With respect to some physical quantities ϕ, the equalization operator in the x-direction is defined as follows. Moreover, the i-th grid point number from the side is represented as the subindex.

$$\overline{\phi}_i^x = \frac{1}{2}\left(\phi_{i-\frac{1}{2}} + \phi_{i+\frac{1}{2}}\right) \tag{6.5}$$

Furthermore, the average of four grid points in the 4th order central difference is defined as follows.

$$\overline{\phi}_i^{2x} = \frac{1}{4}\left(\phi_{i-\frac{3}{2}} + \phi_{i-\frac{1}{2}} + \phi_{i+\frac{1}{2}} + \phi_{i+\frac{3}{2}}\right) \tag{6.6}$$

In the case with two indexes of equalization, the equalization operator for x, z is defined as follows (The indexes of grid point number are represented as i and k in the x-direction and the z-direction, respectively.).

$$\overline{\phi}_{ik}^{xz} = \frac{1}{4}\left(\phi_{i-\frac{1}{2},k-\frac{1}{2}} + \phi_{i+\frac{1}{2},k-\frac{1}{2}} + \phi_{i-\frac{1}{2},k+\frac{1}{2}} + \phi_{i+\frac{1}{2},k+\frac{1}{2}}\right) \tag{6.7}$$

Moreover, the differential operator is defined by representing the grid interval as Δx in x-direction as follows. The i-th grid point number from the side is also represented as the subindex.

$$(\partial_x \phi)_i = \frac{1}{\Delta x}\left(\phi_{i+\frac{\Delta x}{2}} - \phi_{i-\frac{\Delta x}{2}}\right) \tag{6.8}$$

$$(\partial_{2x} \phi)_i = \frac{1}{2\Delta x}\left(\phi_{i+\Delta x} - \phi_{i-\Delta x}\right) \tag{6.9}$$

Dependent variables in the numerical computing are defined by using these values as follows.

$$\rho^* = \bar{\rho} G^{\frac{1}{2}} \tag{6.10}$$

$$u^* = \overline{\bar{\rho} G^{\frac{1}{2}}}^x u \tag{6.11}$$

$$v^* = \overline{\bar{\rho} G^{\frac{1}{2}}}^y v \tag{6.12}$$

$$w^* = \overline{\bar{\rho} G^{\frac{1}{2}}}^\zeta w \tag{6.13}$$

$$W^* = \overline{\bar{\rho} G^{\frac{1}{2}}}^\zeta W \tag{6.14}$$

Here, the density $\bar{\rho}$ and the Jacobian $G^{\frac{1}{2}}$ for coordinate transformation are defined at the same point as the scalars. The vertical velocity W in a terrain-following coordinates and the vertical velocity w in the Cartesian coordinates are defined at the same point.

$$W = \frac{\left(\overline{u^\zeta J_{31}}^x + \overline{v^\zeta J_{32}}^y + w\right)}{\overline{G^{\frac{1}{2}}}^\zeta} \tag{6.15}$$

In the case of explicit scheme, descretization of the equations of (6.1) $\sim$ (??) are given with respect to u, v, w, p' as follows.

$$\overline{\rho^*}^x \frac{u^{\tau+\Delta\tau} - u^\tau}{\Delta\tau} = -\left[\partial_x\left(J_d P_\alpha\right) + \partial_\zeta\left(J_{31}\overline{P_\alpha}^{\zeta x}\right)\right]^\tau + F_u^t \tag{6.16}$$

$$\overline{\rho^*}^y \frac{v^{\tau+\Delta\tau} - v^\tau}{\Delta\tau} = -\left[\partial_y\left(J_d P_\alpha\right) + \partial_\zeta\left(J_{32}\overline{P_\alpha}^{\zeta y}\right)\right]^\tau + F_v^t \tag{6.17}$$

$$\overline{\rho^*}^\zeta \frac{w^{\tau+\Delta\tau} - w^\tau}{\Delta\tau} = -\left[\partial_\zeta P_\alpha\right]^\tau - \left[\overline{gG^{\frac{1}{2}}\frac{p'}{c_s^2}}^\zeta\right]^\tau + F_w^t \tag{6.18}$$

$$\overline{G^{\frac{1}{2}}}^\zeta \frac{p'^{\tau+\Delta\tau} - p'^\tau}{\Delta\tau} = -\bar{\rho}c_s^2\left[\partial_x\left(\overline{G^{\frac{1}{2}}}^x u\right) + \partial_\zeta\left(\overline{J_{31}\overline{u}^\zeta}^x\right) + \partial_y\left(\overline{G^{\frac{1}{2}}}^y v\right) + \partial_\zeta\left(\overline{J_{32}\overline{v}^\zeta}^y\right) + \partial_\zeta w\right]^{\tau+\Delta\tau}$$
$$+ g\rho^*\left[\overline{w}^\zeta\right]^{\tau+\Delta\tau} + F_p^t \tag{6.19}$$

where

$$P_\alpha = p' - \alpha Div^* \tag{6.20}$$

$$Div^* = \frac{1}{G^{\frac{1}{2}}}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial W^*}{\partial \zeta}\right) \tag{6.21}$$

where α is a factor of damping for divergence terms to weaken the effects of the sound wave. The terms concerning the sound wave, which are integrated with a small time step $\Delta\tau$, are explicitly written, the other terms are included in F_ϕ^t ($\phi = u, v, w, p'$). The first term on the right side in the equations of (6.16) and (6.17), and the second term on the right side in the equation of (6.18) are pressure gradient force term. Buoyancy force with pressure perturbations and divergence damping are shown by the first and third term on the right side in the equation (6.18), respectively. The first term on the right side in the equation of (6.19) is a divergence term and the second term of that is a vertical advection term.

When terms for sound wave are solved explicitly, u, v, and w are calculated by the equations of (6.16) $\sim$ (6.18) with forward-difference for $\Delta\tau$. Using the values of u, v, and w, pressure perturbation p' is calculated by the equation of (6.19) with backward-difference for $\Delta\tau$. During this integration with a small time step for $2\Delta\tau$ ($= n\Delta\tau$), F_ϕ^t is constant. Terms of the sound wave in most of non-hydrostatic models are not solved explicitly in vertical and horizontal. Although some models (e.g. ARPS) include the HE-VE scheme as an option, the NHM of Meteorological Research Institute adopts the HI-VI and HE-VI schemes. The technique to solve inplicitly in only vertical is described later. However, it is not necessarily that terms for sound wave are calculated inplicitly in vertical when the horizontal resolution is the same as the verical resolution as an object of **CReSS**. There is a option to calculate the gravity mode in the small time step in **CReSS**. This technique is also shown later. The gravity wave mode is integrated in the small time step in ARPS and in the large time step in NHM.

In the case of using vertical implicit scheme, the equations of verical velocity w and pressure perturbation p' in the equations (6.18 and 6.19) are replaced by the following equations.

$$\overline{\rho^*}^\zeta \frac{w^{\tau+\Delta\tau} - w^\tau}{\Delta\tau} = [\partial_\zeta(\alpha Div^*)]^\tau - \left[\beta\,(\partial_\zeta p')^{\tau+\Delta\tau} + (1-\beta)\,(\partial_\zeta p')^\tau\right]$$
$$- \left[\beta\left(\overline{gG^{\frac{1}{2}}\frac{p'}{c_s^2}}^\zeta\right)^{\tau+\Delta\tau} + (1-\beta)\left(\overline{gG^{\frac{1}{2}}\frac{p'}{c_s^2}}^\zeta\right)^\tau\right] + F_w^t \tag{6.22}$$

$$\overline{G^{\frac{1}{2}}}^\zeta \frac{p'^{\tau+\Delta\tau} - p'^\tau}{\Delta\tau} = -\bar{\rho}c_s^2 \left[\partial_x\left(\overline{G^{\frac{1}{2}}}^x u\right) + \partial_\zeta\left(\overline{J_{31}\overline{u}^\zeta}^x\right) + \partial_y\left(\overline{G^{\frac{1}{2}}}^y v\right) + \partial_\zeta\left(\overline{J_{32}\overline{v}^\zeta}^y\right)\right]^{\tau+\Delta\tau}$$
$$-\bar{\rho}c_s^2 \left[\beta\,(\partial_\zeta w)^{\tau+\Delta\tau} + (1-\beta)\,(\partial_\zeta w)^\tau\right]$$
$$+g\rho^* \left[\beta\left(\overline{w}^\zeta\right)^{\tau+\Delta\tau} + (1-\beta)\left(\overline{w}^\zeta\right)^\tau\right] + F_p^t \tag{6.23}$$

These equations are gained by averaging for τ and $\tau + \Delta\tau$ of the equations (6.18) and (6.19) weighted by β. The equations of w (6.22) with $\beta = 0$ and p' (6.23) with $\beta = 1$ result in the equations of (6.18) and (6.19), respectively. When $\beta = 0$, (6.22) and (6.23) are teh simultaneous equations for $w^{\tau+\Delta\tau}$ and $p^{\tau+\Delta\tau}$. They are implicitly calculated in vertical wtih the Crank-Nicolson scheme.

In the vertical implicit scheme, the forward difference for the equations of (6.16) and (6.17) are calculated and the values of u and v at $\tau + \Delta\tau$ are gained. In the vertical implicit scheme, the forward difference for the equations of (6.16) and (6.17) are calculated and the values of u and v at $\tau + \Delta\tau$ are gained. Concerning $p'^{\tau+\Delta\tau}$ from (6.22) and (6.23), we get the following equations for w and p.

$$\overline{\rho^*}^\zeta \frac{w^{\tau+\Delta\tau} - w^\tau}{\Delta\tau} = -\Delta\tau\beta^2\partial_\zeta\left(\frac{g\rho^*}{G^{\frac{1}{2}}}\overline{w}^\zeta - \frac{\bar{\rho}c_s^2}{G^{\frac{1}{2}}}\partial_\zeta w\right)^{\tau+\Delta\tau}$$
$$-\Delta\tau\beta^2 g\left(\overline{\frac{g\rho^*}{c_s^2}\overline{w}^\zeta} - \overline{\bar{\rho}\partial_\zeta w}^\zeta\right)^{\tau+\Delta\tau} + F_w' \tag{6.24}$$

where F_w' and F_p' are

$$F_p' = \frac{\Delta\tau}{G^{\frac{1}{2}}}\left[F_p^t + (1-\beta)\left(g\rho^*\overline{w}^\zeta - \bar{\rho}c_s^2\partial_\zeta w\right)^\tau\right]$$
$$-\frac{\Delta\tau}{G^{\frac{1}{2}}}\bar{\rho}c_s^2\left[\partial_x\left(\overline{G^{\frac{1}{2}}}^x u\right) + \partial_\zeta\left(\overline{J_{31}\overline{u}^\zeta}^x\right) + \partial_y\left(\overline{G^{\frac{1}{2}}}^y v\right) + \partial_\zeta\left(\overline{J_{32}\overline{v}^\zeta}^y\right)\right]^{\tau+\Delta\tau}$$
$$\tag{6.25}$$

$$F_w' = -[\partial_\zeta P_\alpha]^\tau - \left[\overline{gG^{\frac{1}{2}}\frac{p'}{c_s^2}}^\zeta\right]^\tau - \beta\left[\overline{\frac{gG^{\frac{1}{2}}F_p'}{c_s^2}}^\zeta + \partial_\zeta F_p'\right] + F_w^t. \tag{6.26}$$

The second-order discretization in vertical (6.24) is ordered as follows for the unknown $w^{\tau+\Delta\tau}$.

$$A_k w_{k-1}^{\tau+\Delta\tau} + B_k w_k^{\tau+\Delta\tau} + C_k w_{k+1}^{\tau+\Delta\tau} = F_k \tag{6.27}$$

Simultaneous equations whose factors are a triple diagonal matrix are obtained as this equation. This equation is solved simply if the top and bottom boundary conditions are given as follows, for example.

$$w = 0 \qquad\qquad\qquad k = nk - 1 \qquad\qquad (6.28)$$

$$w = u\,\frac{\partial z_s}{\partial x} + v\frac{\partial z_s}{\partial y} \qquad k = 2 \qquad\qquad (6.29)$$

where A_k, B_k, C_k, F_k are known quantities at τ. These quantities are given under the boundary conditions of (6.28) and (6.29) as follows.

$$A_k = \begin{cases} 0 & k = 3 \\[2mm] \left(-\mathcal{Q}_k c_{s\,k-1}^2 + \mathcal{P}_k G_{k-1}^{\frac{1}{2}}\right)(\mathcal{R}_{k-1} + \mathcal{S}_{k-1}), & 4 \le k \le nk - 2 \end{cases} \qquad (6.30)$$

$$\begin{aligned} B_k = 1 + \mathcal{Q}_k\left\{(\mathcal{R}_k + \mathcal{S}_k)\,c_{s\,k}^2 - (\mathcal{R}_{k-1} - \mathcal{S}_{k-1})\,c_{s\,k-1}^2\right\} \\ + \mathcal{P}_k\left\{(\mathcal{R}_k + \mathcal{S}_k)\,G_k^{\frac{1}{2}} - (\mathcal{R}_{k-1} - \mathcal{S}_{k-1})\,G_{k-1}^{\frac{1}{2}}\right\}, \quad 3 \le k \le nk - 2 \end{aligned} \qquad (6.31)$$

$$C_k = \begin{cases} \left(\mathcal{Q}_k c_{s\,k}^2 + \mathcal{P}_k G_k^{\frac{1}{2}}\right)(\mathcal{R}_k + \mathcal{S}_k), & 3 \le k \le nk - 3 \\[2mm] 0, & k = nk - 2 \end{cases} \qquad (6.32)$$

$$F_k = \begin{cases} F'_{w\,k} + w_k^\tau + \overline{u}_k^\zeta\,(\partial_x z_s)_k + \overline{v}_k^\zeta\,(\partial_y z_s)_k, & k = 3 \\[2mm] F'_{w\,k} + w_k^\tau, & 4 \le k \le nk - 2 \end{cases} \qquad (6.33)$$

$$\mathcal{P}_k = \frac{\Delta\tau^2\beta^2 g}{2\overline{\rho_k^*}^\zeta}, \quad \mathcal{Q}_k = \frac{\Delta\tau^2\beta^2}{\Delta\zeta_k\overline{\rho_k^*}^\zeta}, \quad \mathcal{R}_k = \frac{g\rho_k^*}{2G_k^{\frac{1}{2}}c_{s\,k}^2}, \quad \mathcal{S}_k = \frac{\rho_k^*}{\Delta\zeta_k G_k} \qquad (6.34)$$

This simultaneous equation (6.27) is represented by a matrix by replacing $w_k^{\tau+\Delta\tau}$ with w_k as follows.

$$
\begin{pmatrix}
B_3 & C_3 & 0 & \cdots\cdots & \cdots & & 0 \\
A_4 & B_4 & C_4 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & A_k & B_k & C_k & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots\cdots & \cdots\cdots & A_{nk-2} & B_{nk-2}
\end{pmatrix}
\begin{pmatrix}
w_3 \\ w_4 \\ \vdots \\ w_k \\ \vdots \\ w_{nk-2}
\end{pmatrix}
=
\begin{pmatrix}
F_3 \\ F_4 \\ \vdots \\ F_k \\ \vdots \\ F_{nk-2}
\end{pmatrix}
\tag{6.35}
$$

The forward elimination and the backward substitution are carried out to solve this matrix by the Gaussian Elimination. The first line in (6.35) are divided by B_3 to carry out the forward elimination.

$$
\begin{pmatrix}
1 & c_3 & 0 & \cdots\cdots & \cdots & & 0 \\
A_4 & B_4 & C_4 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & A_k & B_k & C_k & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots\cdots & \cdots\cdots & A_{nk-2} & B_{nk-2}
\end{pmatrix}
\begin{pmatrix}
w_3 \\ w_4 \\ \vdots \\ w_k \\ \vdots \\ w_{nk-2}
\end{pmatrix}
=
\begin{pmatrix}
f_3 \\ F_4 \\ \vdots \\ F_k \\ \vdots \\ F_{nk-2}
\end{pmatrix}
\tag{6.36}
$$

where $c_3 = C_3/B_3$ and $f_3 = F_3/B_3$. The diagonal components of the values which subtracted from the second line to the values which multiplied the first line by A_4 are standardized as follows.

$$
\begin{pmatrix}
1 & c_3 & 0 & \cdots\cdots & \cdots & & 0 \\
0 & 1 & c_4 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & A_k & B_k & C_k & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots\cdots & \cdots\cdots & A_{nk-2} & B_{nk-2}
\end{pmatrix}
\begin{pmatrix}
w_3 \\ w_4 \\ \vdots \\ w_k \\ \vdots \\ w_{nk-2}
\end{pmatrix}
=
\begin{pmatrix}
f_3 \\ f_4 \\ \vdots \\ F_k \\ \vdots \\ F_{nk-2}
\end{pmatrix}
\tag{6.37}
$$

where $c_4 = C_4/(B_4 - A_4 c_3)$ and $f_4 = (F_4 - A_4 f_3)/(B_4 - A_4 c_3)$. Similarly, using $c_k = C_k/(B_k - A_k c_{k-1})$ and $f_k = (F_k - A_k f_{k-1})/(B_k - A_k c_{k-1})$,

$$
\begin{pmatrix}
1 & c_3 & 0 & \cdots\cdots\cdots & 0 \\
0 & 1 & c_4 & 0 & \cdots\cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & c_k & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots\cdots\cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
w_3 \\ w_4 \\ \vdots \\ w_k \\ \vdots \\ w_{nk-2}
\end{pmatrix}
=
\begin{pmatrix}
f_3 \\ f_4 \\ \vdots \\ f_k \\ \vdots \\ f_{nk-2}
\end{pmatrix}
\tag{6.38}
$$

Finally, solutions for all k are obtained to carry out the backward substitution in order of $k = nk-2, nk-3$ and $\cdots, 4$.

$$w_{nk-2} = f_{nk-2}$$
$$w_{nk-3} = f_{nk-3} - c_{nk-3}w_{nk-2}$$
$$\vdots$$
$$w_k \quad = f_k - c_k w_{k+1} \tag{6.39}$$
$$\vdots$$
$$w_4 \quad = f_4 - c_4 w_5$$
$$w_3 \quad = f_3 - c_3 w_4$$

The term (F_ϕ^t) which is calculated with the large time step is represented as follows.

$$F_u^t = -\mathrm{Adv}.u^t + \left[\overline{\rho^* f_s \overline{v}^y}^x - \overline{\rho^* f_c \overline{w}}^\varsigma\right]^t + \left[G^{\frac{1}{2}}\mathrm{Turb}.u\right]^{t-\Delta t} \tag{6.40}$$

$$F_v^t = -\mathrm{Adv}.v^t - \left[\overline{\rho^* f_s \overline{u}^x}^y\right]^t + \left[G^{\frac{1}{2}}\mathrm{Turb}.v\right]^{t-\Delta t} \tag{6.41}$$

$$F_w^t = -\mathrm{Adv}.w^t + \left[\overline{\rho^* \left(\mathrm{Buoy}.\theta + \mathrm{Buoy}.q\right)}^\varsigma + \overline{\rho^* f_c \overline{u}^x}^\varsigma\right]^t + \left[G^{\frac{1}{2}}\mathrm{Turb}.w\right]^{t-\Delta t} \tag{6.42}$$

$$F_p^t = -\mathrm{Adv}.p^t \tag{6.43}$$

where Buoy.θ and Buoy.q are buoyancy terms for temperature perturbations, watar vapor and hydrometeors, and are defined as follows.

$$\mathrm{Buoy}.\theta \equiv g\frac{\theta'}{\bar{\theta}} \tag{6.44}$$

$$\mathrm{Buoy}.q \equiv g\left(\frac{q'_v}{\epsilon + \bar{q}_v} - \frac{q'_v + \sum q_x}{1 + \bar{q}_v}\right) \tag{6.45}$$

The advection term is calculated with the second or fourth-order accuracy by using the values at the time (t). The second-order advection term is descretized as follows.

$$\mathrm{Adv}.u^t = \overline{\overline{u^{*x}}\partial_x u}^x + \overline{\overline{v^{*x}}\partial_y u}^y + \overline{\overline{W^{*x}}\partial_\varsigma u}^\varsigma \tag{6.46}$$

$$\mathrm{Adv}.v^t = \overline{\overline{u^{*y}}\partial_x v}^x + \overline{\overline{v^{*y}}\partial_y v}^y + \overline{\overline{W^{*y}}\partial_\varsigma v}^\varsigma \tag{6.47}$$

$$\mathrm{Adv}.w^t = \overline{\overline{u^{*\varsigma}}\partial_x w}^x + \overline{\overline{v^{*\varsigma}}\partial_y w}^y + \overline{\overline{W^{*\varsigma}}\partial_\varsigma v}^\varsigma \tag{6.48}$$

$$\mathrm{Adv}.p^t = \overline{\overline{G^{\frac{1}{2}}}^x u\partial_x p'}^x + \overline{\overline{G^{\frac{1}{2}}}^y v\partial_y p'}^y + \overline{\overline{G^{\frac{1}{2}}}^y W\partial_\varsigma p'}^\varsigma \tag{6.49}$$

The fourth-order advection term is descretized as follows.

$$\mathrm{Adv}.u^t = \frac{4}{3}\left[\overline{\overline{u^{*x}}\partial_x u}^x + \overline{\overline{v^{*x}}\partial_y u}^y + \overline{\overline{W^{*x}}\partial_\varsigma u}^\varsigma\right]$$
$$- \frac{1}{3}\left[\overline{\overline{u^{*2x}}\partial_{2x} u}^{2x} + \overline{\overline{v^{*xy}}\partial_{2y} u}^{2y} + \overline{\overline{W^{*x\varsigma}}\partial_{2\varsigma} u}^{2\varsigma}\right] \tag{6.50}$$

$$\text{Adv.}v^t = \frac{4}{3}\left[\overline{\overline{u^{*y}}\partial_x v}^x + \overline{\overline{v^{*y}}\partial_y v}^y + \overline{\overline{W^{*y}}\partial_\zeta v}^\zeta\right]$$
$$-\frac{1}{3}\left[\overline{\overline{u^{*xy}}\partial_{2x}v}^{2x} + \overline{\overline{v^{*2y}}\partial_{2y}v}^{2y} + \overline{\overline{W^{*y\zeta}}\partial_{2\zeta}v}^{2\zeta}\right] \tag{6.51}$$

$$\text{Adv.}w^t = \frac{4}{3}\left[\overline{\overline{u^{*\zeta}}\partial_x w}^x + \overline{\overline{v^{*\zeta}}\partial_y w}^y + \overline{\overline{W^{*\zeta}}\partial_\zeta v}^\zeta\right]$$
$$-\frac{1}{3}\left[\overline{\overline{u^{*x\zeta}}\partial_{2x}w}^{2x} + \overline{\overline{v^{*y\zeta}}\partial_{2y}w}^{2y} + \overline{\overline{W^{*2\zeta}}\partial_{2\zeta}v}^{2\zeta}\right] \tag{6.52}$$

$$\text{Adv.}p^t = \frac{4}{3}\left[\overline{\overline{G^{\frac{1}{2}}}^x u\partial_x p'}^x + \overline{\overline{G^{\frac{1}{2}}}^y v\partial_y p'}^y + \overline{\overline{G^{\frac{1}{2}}}^y W\partial_\zeta p'}^\zeta\right]$$
$$-\frac{1}{3}\left[\overline{\overline{\overline{G^{\frac{1}{2}}}^x}^x u\ \partial_{2x}p'}^{2x} + \overline{\overline{\overline{G^{\frac{1}{2}}}^y}^y v\ \partial_{2y}p'}^{2y} + \overline{\overline{\overline{G^{\frac{1}{2}}}^\zeta}^\zeta W\ \partial_{2\zeta}p'}^{2\zeta}\right] \tag{6.53}$$

The computing instability is caused in the calculation with discretization of central difference. So an artificial numerical viscosity term which is shown in the following section 6.2 is added to the above terms.

Variables except for pressure in a turbulent diffusion term are applied in the turbulent diffsion terms. The tensor for deformation velocity which are indicated in (??) ∼ (??) are solved as follows by using calculus of finite differences.

$$S_{11} = \frac{2}{G^{\frac{1}{2}}}\left[\partial_x\left(\overline{J_d}^x u\right) + \partial_\zeta\left(\overline{J_{31}\overline{u}^\zeta}^x\right)\right] \tag{6.54}$$

$$S_{22} = \frac{2}{G^{\frac{1}{2}}}\left[\partial_y\left(\overline{J_d}^y v\right) + \partial_\zeta\left(\overline{J_{32}\overline{v}^\zeta}^y\right)\right] \tag{6.55}$$

$$S_{33} = \frac{2}{G^{\frac{1}{2}}}\partial_\zeta w \tag{6.56}$$

$$S_{12} = \frac{1}{\overline{G^{\frac{1}{2}}}^{xy}}\left[\partial_y\left(\overline{J_d}^x u\right) + \partial_x\left(\overline{J_d}^y v\right) + \partial_\zeta\left(\overline{J_{32}}^x \overline{u}^{\zeta y} + \overline{J_{31}}^y \overline{v}^{\zeta x}\right)\right] \tag{6.57}$$

$$S_{13} = \frac{1}{\overline{G^{\frac{1}{2}}}^{x\zeta}}\left[\partial_x\left(\overline{J_d}^\zeta w\right) + \partial_\zeta\left(u + \overline{J_{31}\overline{w}^x}^\zeta\right)\right] \tag{6.58}$$

$$S_{23} = \frac{1}{\overline{G^{\frac{1}{2}}}^{y\zeta}}\left[\partial_y\left(\overline{J_d}^\zeta w\right) + \partial_\zeta\left(v + \overline{J_{32}\overline{w}^y}^\zeta\right)\right] \tag{6.59}$$

The divergence term Div which is indicated in 3.24 are solved as follows by using calculus of finite differences.

$$Div = \frac{1}{G^{\frac{1}{2}}}\left[\partial_x\left(\overline{G^{\frac{1}{2}}}^x u\right) + \partial_y\left(\overline{G^{\frac{1}{2}}}^y v\right) + \partial_\zeta\left(\overline{G^{\frac{1}{2}}}^\zeta W\right)\right] \tag{6.60}$$

Using these, the stress tensor which are indicated in (??) ∼ (??) are written by finit differences as follows.

$$\tau_{11} = \bar{\rho}\nu_{\tau h}\left(S_{11} - \frac{2}{3}Div\right) \tag{6.61}$$

$$\tau_{12} = \overline{\bar{\rho}\nu_{\tau h}}^{xy}S_{12} \tag{6.62}$$

$$\tau_{13} = \overline{\bar{\rho}\nu_{\tau v}}^{x\zeta}S_{13} \tag{6.63}$$

$$\tau_{21} = \overline{\bar{\rho}\nu_{\tau h}}^{yx}S_{21} \tag{6.64}$$

$$\tau_{22} = \bar{\rho}\nu_{\tau h}\left(S_{22} - \frac{2}{3}Div\right) \tag{6.65}$$

$$\tau_{23} = \overline{\bar{\rho}\nu_{\tau v}}^{y\zeta}S_{23} \tag{6.66}$$

$$\tau_{31} = \overline{\bar{\rho}\nu_{\tau h}}^{\zeta x}S_{31} \tag{6.67}$$

$$\tau_{32} = \overline{\bar{\rho}\nu_{\tau h}}^{\zeta y}S_{32} \tag{6.68}$$

$$\tau_{33} = \bar{\rho}\nu_{\tau v}\left(S_{33} - \frac{2}{3}Div\right) \tag{6.69}$$

Finally, the turbulent diffusion terms are indicated in differences as follows.

$$G^{\frac{1}{2}}\mathrm{Turb}.u = \partial_x\left(J_d\tau_{11}\right) + \partial_y\left(\overline{J_d}^{xy}\tau_{12}\right) + \partial_\zeta\left(\tau_{13} + J_{31}\overline{\tau_{11}}^{x\zeta} + \overline{\overline{J_{32}}^x\overline{\tau_{12}}^\zeta}^{y}\right) \tag{6.70}$$

$$G^{\frac{1}{2}}\mathrm{Turb}.v = \partial_x\left(\overline{J_d}^{yx}\tau_{21}\right) + \partial_y\left(J_d\tau_{22}\right) + \partial_\zeta\left(\tau_{23} + \overline{\overline{J_{31}}^y\overline{\tau_{21}}^\zeta}^{x} + J_{32}\overline{\tau_{22}}^{y\zeta}\right) \tag{6.71}$$

$$G^{\frac{1}{2}}\mathrm{Turb}.w = \partial_x\left(\overline{J_d}^{\zeta x}\tau_{31}\right) + \partial_y\left(\overline{J_d}^{\zeta y}\tau_{32}\right) + \partial_\zeta\left(\tau_{33} + \overline{\overline{J_{31}}^\zeta\overline{\tau_{31}}^\zeta}^{x} + \overline{\overline{J_{32}}^\zeta\overline{\tau_{32}}^\zeta}^{y}\right) \tag{6.72}$$

The method of discretization for physical quanities without the sound wave mode

When the gravity wave mode is calculated with the large time step, the equations of potential temperature, water vapor and hydrometeors are solved with only large time step. These equations are shown by indicating water vapor and hydrometeors as x as follows.

$$\rho^* \frac{\theta'^{t+\Delta t} - \theta'^{t-\Delta t}}{2\Delta t} = F_\theta^t \tag{6.73}$$

$$\rho^* \frac{q_x^{t+\Delta t} - q_x^{t-\Delta t}}{2\Delta t} = F_q^t \tag{6.74}$$

$$G^{\frac{1}{2}} \frac{N_x^{t+\Delta t} - N_x^{t-\Delta t}}{2\Delta t} = F_N^t \tag{6.75}$$

where

$$F_\theta^t = -\text{Adv.}\theta^t + \left[G^{\frac{1}{2}}\text{Turb.}\theta \right]^{t-\Delta t} + [\rho^*\text{Src.}\theta]^t - \left[\overline{\overline{\rho}^\zeta w \partial_\zeta \overline{\theta}}^\zeta \right]^t \tag{6.76}$$

$$F_q^t = -\text{Adv.}q_x^t + \left[G^{\frac{1}{2}}\text{Turb.}q_x \right]^{t-\Delta t} + [\rho^*\text{Src.}q_x]^t + [\rho^*\text{Fall.}q_x]^t \tag{6.77}$$

$$F_N^t = -\text{Adv.}\frac{N_x}{\rho^*}^t + \left[G^{\frac{1}{2}}\text{Turb.}\frac{N_x}{\rho^*} \right]^{t-\Delta t} + \left[\rho^*\text{Src.}\frac{N_x}{\rho^*} \right]^t + \left[\rho^*\text{Fall.}\frac{N_x}{\rho^*} \right]^t. \tag{6.78}$$

The fourth term on the right side of the equation of potential temperature perturbation (6.76) is The term of gravity wave with vertical advection of potential temperature in the basic state.

The advection term is calculated as physical quanities with the sound wave mode by using the values at the time t. The advection term is solved by using the second-order scheme as follows.

$$\text{Adv.}\theta^t = \overline{u^*\partial_x\theta}^x + \overline{v^*\partial_y\theta}^y + \overline{W^*\partial_\zeta\theta}^\zeta \tag{6.79}$$

$$\text{Adv.}q_x^t = \overline{u^*\partial_x q_x}^x + \overline{v^*\partial_y q_x}^y + \overline{W^*\partial_\zeta q_x}^\zeta \tag{6.80}$$

$$\text{Adv.}\frac{N_x}{\rho^*}^t = \overline{u^*\partial_x\frac{N_x}{\rho^*}}^x + \overline{v^*\partial_y\frac{N_x}{\rho^*}}^y + \overline{W^*\partial_\zeta\frac{N_x}{\rho^*}}^\zeta \tag{6.81}$$

The advection terms are solved by using the fourth-order scheme as follows.

$$\begin{aligned}
\text{Adv.}\theta'^t = &\frac{4}{3}\left[\overline{u^*\partial_x\theta}^x + \overline{v^*\partial_y\theta}^y + \overline{W^*\partial_\zeta\theta}^\zeta \right] \\
&- \frac{1}{3}\left[\overline{\overline{u^*}^x\partial_{2x}\theta'}^{2x} + \overline{\overline{v^*}^y\partial_{2y}\theta'}^{2y} + \overline{\overline{W^*}^\zeta\partial_{2\zeta}\theta'}^{2\zeta} \right]
\end{aligned} \tag{6.82}$$

$$\begin{aligned}
\text{Adv.}q_x^t = &\frac{4}{3}\left[\overline{u^*\partial_x q_x}^x + \overline{v^*\partial_y q_x}^y + \overline{W^*\partial_\zeta q_x}^\zeta \right] \\
&- \frac{1}{3}\left[\overline{\overline{u^*}^x\partial_{2x}q_x}^{2x} + \overline{\overline{v^*}^y\partial_{2y}q_x}^{2y} + \overline{\overline{W^*}^\zeta\partial_{2\zeta}q_x}^{2\zeta} \right]
\end{aligned} \tag{6.83}$$

$$\text{Adv.}\frac{N_x}{\rho^*}^t = \frac{4}{3}\left[\overline{u^*\partial_x\frac{N_x}{\rho^*}}^x + \overline{v^*\partial_y\frac{N_x}{\rho^*}}^y + \overline{W^*\partial_\zeta\frac{N_x}{\rho^*}}^\zeta \right]$$

$$-\frac{1}{3}\left[\overline{\overline{u^*}^x\partial_{2x}\frac{N_x}{\rho^*}}^{2x}+\overline{\overline{v^*}^y\partial_{2y}\frac{N_x}{\rho^*}}^{2y}+\overline{\overline{W^*}^\zeta\partial_{2\zeta}\frac{N_x}{\rho^*}}^{2\zeta}\right] \tag{6.84}$$

The turbulent flux $H_{\phi 1}, H_{\phi 2}$, and $H_{\phi 3}$ for one scalar ϕ which is represented by (??) $\sim$ (??) are solved by using difference as follows.

$$H_{\phi 1}=\overline{\left(\bar\rho\nu_{Hh}\frac{1}{G^{\frac{1}{2}}}\right)}^x\left[\partial_x\left(J_d\phi\right)+\partial_\zeta\left(J_{31}\overline{\phi}^{x\zeta}\right)\right] \tag{6.85}$$

$$H_{\phi 2}=\overline{\left(\bar\rho\nu_{Hh}\frac{1}{G^{\frac{1}{2}}}\right)}^y\left[\partial_y\left(J_d\phi\right)+\partial_\zeta\left(J_{32}\overline{\phi}^{y\zeta}\right)\right] \tag{6.86}$$

$$H_{\phi 3}=\overline{\left(\bar\rho\nu_{Hv}\frac{1}{G^{\frac{1}{2}}}\right)}^\zeta\partial_\zeta\overline{\phi}^\zeta \tag{6.87}$$

Finally, the turbulent diffsion term is indicated as follows.

$$G^{\frac{1}{2}}\mathrm{Turb}.\theta=\partial_x\left(\overline{J_d}^xH_{\theta 1}\right)+\partial_y\left(\overline{J_d}^yH_{\theta 2}\right)+\partial_\zeta\left(H_{\theta 3}+\overline{J_{31}\overline{H_{\theta 1}}^\zeta}^x+\overline{J_{32}\overline{H_{\theta 2}}^\zeta}^y\right) \tag{6.88}$$

$$G^{\frac{1}{2}}\mathrm{Turb}.q_x=\partial_x\left(\overline{J_d}^xH_{q_x1}\right)+\partial_y\left(\overline{J_d}^yH_{q_x2}\right)+\partial_\zeta\left(H_{q_x3}+\overline{J_{31}\overline{H_{q_x1}}^\zeta}^x+\overline{J_{32}\overline{H_{q_x2}}^\zeta}^y\right) \tag{6.89}$$

$$G^{\frac{1}{2}}\mathrm{Turb}.\frac{N_x}{\rho^*}=\partial_x\left(\overline{J_d}^xH_{N_x1}\right)+\partial_y\left(\overline{J_d}^yH_{N_x2}\right)+\partial_\zeta\left(H_{N_x3}+\overline{J_{31}\overline{H_{N_x1}}^\zeta}^x+\overline{J_{32}\overline{H_{N_x2}}^\zeta}^y\right) \tag{6.90}$$

The method of discretization in the case of calculating the gravity wave mode with small time step

When the gravity wave mode is calculated with small time step, the equation of potential temperature is indicated as follows.

$$\frac{\partial\theta^*}{\partial t}=\underbrace{-\left(u^*\frac{\partial\theta'}{\partial x}+v^*\frac{\partial\theta'}{\partial y}+W^*\frac{\partial\theta'}{\partial\zeta}\right)}_{[rm]}\underbrace{-\bar\rho w\frac{\partial\bar\theta}{\partial\zeta}}_{[gm]}+\underbrace{G^{\frac{1}{2}}\mathrm{Turb}.\theta}_{[physics]}+\underbrace{\rho^*\mathrm{Src}.\theta}_{[physics]} \tag{6.91}$$

The second term on the right side of is calculated by using the values at the time step $\Delta\tau$. The time integration of this equation is shown as follows.

$$\rho^*\frac{\theta'^{\tau+\Delta\tau}-\theta'^\tau}{\Delta\tau}=-\left[\overline{\overline{\bar\rho}^\zeta w\partial_\zeta\bar\theta}^\zeta\right]^\tau+F_\theta^t \tag{6.92}$$

The time integration of vertical velocity (w) is shown since the buoyancy terms are calculated by using the values at the time$(\Delta \tau)$.

$$\overline{\rho^*}^\zeta \frac{w^{\tau + \Delta \tau} - w^\tau}{\Delta \tau} = [\partial_\zeta (\alpha Div^*)]^\tau + g \left[\overline{\rho^* \frac{\theta'}{\overline{\theta}}}^\zeta \right]^\tau - \left[\beta (\partial_\zeta p')^{\tau + \Delta \tau} + (1 - \beta) (\partial_\zeta p')^\tau \right]$$

$$- \left[\beta \left(\overline{gG^{\frac{1}{2}} \frac{p'}{c_s^2}}^\zeta \right)^{\tau + \Delta \tau} + (1 - \beta) \left(\overline{gG^{\frac{1}{2}} \frac{p'}{c_s^2}}^\zeta \right)^\tau \right] + F_w^t \tag{6.93}$$

Terms which are calculated with large time step are modified as follows by this.

$$F_w^t = -\text{Adv}.w^t + \left[\overline{\rho^*\text{Buoy}.q}^\zeta + \overline{\rho^* f_c \overline{u}^x}^\zeta \right]^t + \left[G^{\frac{1}{2}} \text{Turb}.w \right]^{t - \Delta t} \tag{6.94}$$

$$F_\theta^t = -\text{Adv}.\theta^t + \left[G^{\frac{1}{2}} \text{Turb}.\theta \right]^{t - \Delta t} + [\rho^*\text{Src}.\theta]^t \tag{6.95}$$

time filter

The large time step integration is solved by using the leap-frog scheme. The separation of solutions between the odd time step and the even time step is caused in the case of using this technique. So the Asselin time filter (Asselin, 1972) is used to control the separation of solutions. This is the technique to filter physical quanities at the time t after physical quanities are calculated at the time $t + \Delta t$.

The Asselin time filter is indicated by using a representative variable ψ for forecasted variables $u, v, w, p', \theta', q_x$.

$$\overline{\psi^t} = \psi^t + \mu_a \left(\overline{\psi^{t - \Delta t}} - 2\psi^t + \psi^{t + \delta t} \right) \tag{6.96}$$

where μ_a is a factor of filter, and $\mu_a = 0.1$ is a standard value.

6.2   Artificial numerical viscosity term

6.2.1   Necessity of numerical viscosity term

In previous section, it is shown that advection term is expressed by second-order or fourth-order central difference, and calculation instability will be generated if the artificial numerical viscosity term is not added. The reason for that is shown as follows.

One-dimensional linear wave equation :

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad (c > 0) \tag{6.97}$$

The equation is discretized as follows. Arrangement of shifted grid points is not considered here.

$$\frac{\partial u}{\partial t} = \frac{u_i^{t + \Delta t} - u_i^t}{\Delta t}, \quad \frac{\partial u}{\partial x} = \frac{u_{i + \Delta x}^t - u_{i - \Delta x}^t}{2\Delta x} \tag{6.98}$$

The equation (6.97) is written by

$$u_i^{t+\Delta t} = u_i^t - \frac{c}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(u_{i+\Delta x}^t - u_{i-\Delta x}^t\right) \tag{6.99}$$

From this equation, the value at time $t + \Delta t$ is calculated by using the value at time t. However, when this calculation is performed, this calculation will break even if the relation of Δt and Δx satisfys the CFL condition.

As the other discrete method, we consider first-order windward difference because its calculation can be stable. However, it does not mean higher accuracy for the calculation. If this discretization is applied to an equation (6.97) and solved as follows.

$$u_i^{t+\Delta t} = u_i^t - c\left(\frac{\Delta t}{\Delta x}\right)\left(u_i^t - u_{i-\Delta x}^t\right) \tag{6.100}$$

However, if this is transformed into the form using the equation (6.99) which is the first discretization,

$$u_i^{t+\Delta t} = u_i^t - \frac{c}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(u_{i+\Delta x}^t - u_{i-\Delta x}^t\right) + \frac{c}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(u_{i+\Delta x}^t - 2u_i^t + u_{i-\Delta x}^t\right) \tag{6.101}$$

The added term is difference approximation of space second-degree differentiation $\dfrac{\partial^2 u}{\partial x^2}$, and affects as the diffusion term. Calculation is stably preformed using the first-order forward difference from this reason. In central difference, however, for making the diffusion term, the artificial numerical viscosity term is added to the advection term.

6.2.2   Numerical viscosity with the second-order or fourth-order diffusion term

Since the eventh-degree differentiation generally has the effect of diffusion, the deffision term explained in the previous subsection can change the degree of the differentiation. Here, numerical viscosities with the second-order and the fourth-order diffusion terms are explained.

Firstly, the second-order viscosity term is represented by the arbitrary physical quantities ϕ,

$$G^{\frac{1}{2}}\text{Diff.}\phi = \nu_{2h}\left[\frac{\partial^2\left(\bar{\rho}\phi'\right)}{\partial x^2} + \frac{\partial^2\left(\bar{\rho}\phi'\right)}{\partial y^2}\right] + \nu_{2v}\frac{\partial^2\left(\bar{\rho}\phi'\right)}{\partial\zeta^2}. \tag{6.102}$$

When ϕ are $u, v, w, q_x, N_x/\rho^*$, ϕ' is perturbation from an initial value. When ϕ is θ, ϕ is the perturbation from the basic state.

ν_{2h} and ν_{2v} are viscosity coefficient, and are defined as follows.

$$\nu_{2h} = \frac{\alpha_{2h}\Delta_h^2}{\Delta t} \tag{6.103}$$

$$\nu_{2v} = \frac{\alpha_{2v}\Delta_v^2}{\Delta t} \tag{6.104}$$

where,

$$\Delta_h = (\Delta x \Delta y)^{\frac{1}{2}} \tag{6.105}$$
$$\Delta_v = \Delta\zeta \tag{6.106}$$

In these equations, Δ_h and Δ_v are the horizontal and vertical averaged grid interval in the domain. Moreover, α_{2h} and α_{2v} are the non-dimensional quantities. For a stable calculation, the following requirement needs to be satisfied.

$$\alpha_{2h} \leq \frac{1}{8}, \quad \alpha_{2v} \leq \frac{1}{8} \tag{6.107}$$

In this way, this viscosity term is applied to forecast variables other than pressure perturbation.

We define the second-order differentiation operator as

$$\left(\partial_x^2 \phi\right)_i = \frac{1}{\Delta x}\left(\phi_{i+\Delta x} - 2\phi_i + \phi_{i-\Delta x}\right) \tag{6.108}$$

When $u', v', w', q_x', N_x'/\rho^*$ are made into perturbation from the initial value and θ' is made into perturbation from the basic state, difference notation are shown as follows.

$$G^{\frac{1}{2}}\text{Diff}.u = \nu_{2h}\left[\partial_x^2\left(\overline{\rho}^x u'\right) + \partial_y^2\left(\overline{\rho}^x u'\right)\right] + \nu_{2v}\left[\partial_\zeta^2\left(\overline{\rho}^x u'\right)\right] \tag{6.109}$$

$$G^{\frac{1}{2}}\text{Diff}.v = \nu_{2h}\left[\partial_x^2\left(\overline{\rho}^y v'\right) + \partial_y^2\left(\overline{\rho}^y v'\right)\right] + \nu_{2v}\left[\partial_\zeta^2\left(\overline{\rho}^y v'\right)\right] \tag{6.110}$$

$$G^{\frac{1}{2}}\text{Diff}.w = \nu_{2h}\left[\partial_x^2\left(\overline{\rho}^\zeta w'\right) + \partial_y^2\left(\overline{\rho}^\zeta w'\right)\right] + \nu_{2v}\left[\partial_\zeta^2\left(\overline{\rho}^\zeta w'\right)\right] \tag{6.111}$$

$$G^{\frac{1}{2}}\text{Diff}.\theta = \nu_{2h}\left[\partial_x^2\left(\bar{\rho}\theta'\right) + \partial_y^2\left(\bar{\rho}\theta'\right)\right] + \nu_{2v}\left[\partial_\zeta^2\left(\bar{\rho}\theta'\right)\right] \tag{6.112}$$

$$G^{\frac{1}{2}}\text{Diff}.q_x = \nu_{2h}\left[\partial_x^2\left(\bar{\rho}q_x'\right) + \partial_y^2\left(\bar{\rho}q_x'\right)\right] + \nu_{2v}\left[\partial_\zeta^2\left(\bar{\rho}q_x'\right)\right] \tag{6.113}$$

$$G^{\frac{1}{2}}\text{Diff}.\frac{N_x}{\rho^*} = \nu_{2h}\left[\partial_x^2\left(\bar{\rho}\frac{N_x'}{\rho^*}\right) + \partial_y^2\left(\bar{\rho}\frac{N_x'}{\rho^*}\right)\right] + \nu_{2v}\left[\partial_\zeta^2\left(\bar{\rho}\frac{N_x'}{\rho^*}\right)\right] \tag{6.114}$$

The fourth-order numerical viscosity term is considered as well as the second-order. When ϕ is $u, v, w, q_x, N_x/\rho^*$, ϕ' is perturbation from an initial value. When ϕ is θ, ϕ is the perturbation from the basic state.

$$G^{\frac{1}{2}}\text{Diff}.\phi = -\nu_{4h}\left[\frac{\partial^4\left(\bar{\rho}\phi'\right)}{\partial x^4} + \frac{\partial^4\left(\bar{\rho}\phi'\right)}{\partial y^4}\right] - \nu_{4v}\frac{\partial^4\left(\bar{\rho}\phi'\right)}{\partial\zeta^4} \tag{6.115}$$

ν_{2h} and ν_{2v} are a viscosity coefficient, and are defined as follows.

$$\nu_{4h} = \frac{\alpha_{4h}\Delta_h^4}{\Delta t} \tag{6.116}$$

$$\nu_{4v} = \frac{\alpha_{4v}\Delta_v^4}{\Delta t} \tag{6.117}$$

where α_{2h} and α_{2v} are the non-dimensional quantities, and the values which are often used are as follows.

$$\alpha_{4h} = 0.001, \quad \alpha_{4v} = 0.001. \tag{6.118}$$

As well as the second-order, this viscosity term is applied to forecast variables other than pressure perturbation.

When $u', v', w', q'_x, N'_x/\rho^*$ are made into perturbation from initial value and θ' is made into perturbation from the basic state, difference notation are shown as follows.

$$G^{\frac{1}{2}}\mathrm{Diff}.u = -\nu_{4h}\left[\partial_x^2\left\{\partial_x^2\left(\overline{\rho}^x u'\right)\right\} + \partial_y^2\left\{\partial_y^2\left(\overline{\rho}^x u'\right)\right\}\right] - \nu_{4v}\partial_\zeta^2\left[\partial_\zeta^2\left(\overline{\rho}^x u'\right)\right] \tag{6.119}$$

$$G^{\frac{1}{2}}\mathrm{Diff}.v = -\nu_{4h}\left[\partial_x^2\left\{\partial_x^2\left(\overline{\rho}^y v'\right)\right\} + \partial_y^2\left\{\partial_y^2\left(\overline{\rho}^y v'\right)\right\}\right] - \nu_{4v}\partial_\zeta^2\left[\partial_\zeta^2\left(\overline{\rho}^y v'\right)\right] \tag{6.120}$$

$$G^{\frac{1}{2}}\mathrm{Diff}.w = -\nu_{4h}\left[\partial_x^2\left\{\partial_x^2\left(\overline{\rho}^\zeta w'\right)\right\} + \partial_y^2\left\{\partial_y^2\left(\overline{\rho}^\zeta w'\right)\right\}\right] - \nu_{4v}\partial_\zeta^2\left[\partial_\zeta^2\left(\overline{\rho}^\zeta w'\right)\right] \tag{6.121}$$

$$G^{\frac{1}{2}}\mathrm{Diff}.\theta = -\nu_{4h}\left[\partial_x^2\left\{\partial_x^2\left(\bar{\rho}\theta'\right)\right\} + \partial_y^2\left\{\partial_y^2\left(\bar{\rho}\theta'\right)\right\}\right] - \nu_{4v}\partial_\zeta^2\left[\partial_\zeta^2\left(\bar{\rho}\theta'\right)\right] \tag{6.122}$$

$$G^{\frac{1}{2}}\mathrm{Diff}.q_x = -\nu_{4h}\left[\partial_x^2\left\{\partial_x^2\left(\bar{\rho}q'_x\right)\right\} + \partial_y^2\left\{\partial_y^2\left(\bar{\rho}q'_x\right)\right\}\right] - \nu_{4v}\partial_\zeta^2\left[\partial_\zeta^2\left(\bar{\rho}q'_x\right)\right] \tag{6.123}$$

$$G^{\frac{1}{2}}\mathrm{Diff}.\frac{N_x}{\rho^*} = -\nu_{4h}\left[\partial_x^2\left\{\partial_x^2\left(\bar{\rho}\frac{N'_x}{\rho^*}\right)\right\} + \partial_y^2\left\{\partial_y^2\left(\bar{\rho}\frac{N'_x}{\rho^*}\right)\right\}\right] - \nu_{4v}\partial_\zeta^2\left[\partial_\zeta^2\left(\bar{\rho}\frac{N'_x}{\rho^*}\right)\right] \tag{6.124}$$

It is more desirable to use this numerical viscosity term which has fourth-order diffusion term, since it can distinguish from the diffusion term Turb.ϕ in Section ?? - "subgrid-scale diffusion", and can diffuse the ingredient of high wave number.

As a result, including these numerical viscosity term, the term F_ϕ^t calculated with large time step t shown in (6.40) $\sim$ (6.43), (6.76) and (6.77) are able to be expressed as follows.

$$F_u^t = -\text{Adv}.u^t + \left[\overline{\rho^* f_s \overline{v}^y}^x - \overline{\rho^* f_c \overline{w}^\zeta}^x\right]^t + \left[G^{\frac{1}{2}}\text{Turb}.u + G^{\frac{1}{2}}\text{Diff}.u\right]^{t-\Delta t} \tag{6.125}$$

$$F_v^t = -\text{Adv}.v^t - \left[\overline{\rho^* f_s \overline{u}^x}^y\right]^t + \left[G^{\frac{1}{2}}\text{Turb}.v + G^{\frac{1}{2}}\text{Diff}.v\right]^{t-\Delta t} \tag{6.126}$$

$$F_w^t = -\text{Adv}.w^t + \left[\overline{\rho^* \left(\text{Buoy}.\theta + \text{Buoy}.q\right)}^\zeta + \overline{\rho^* f_c \overline{u}^x}^\zeta\right]^t + \left[G^{\frac{1}{2}}\text{Turb}.w + G^{\frac{1}{2}}\text{Diff}.w\right]^{t-\Delta t} \tag{6.127}$$

$$F_\theta^t = -\text{Adv}.\theta'^t + \left[G^{\frac{1}{2}}\text{Turb}.\theta' + G^{\frac{1}{2}}\text{Diff}.\theta\right]^{t-\Delta t} + [\rho^*\text{Src}.\theta']^t - \left[\overline{\overline{\rho}^\zeta w \partial_\zeta \overline{\theta}}^\zeta\right]^t \tag{6.128}$$

$$F_q^t = -\text{Adv}.q_x^t + \left[G^{\frac{1}{2}}\text{Turb}.q_x + G^{\frac{1}{2}}\text{Diff}.q_x\right]^{t-\Delta t} + [\rho^*\text{Src}.q_x]^t + [\rho^*\text{Fall}.q_x]^t \tag{6.129}$$

$$F_N^t = -\text{Adv}.\frac{N_x}{\rho^*}^t + \left[G^{\frac{1}{2}}\text{Turb}.\frac{N_x}{\rho^*} + G^{\frac{1}{2}}\text{Diff}.\frac{N_x}{\rho^*}\right]^{t-\Delta t} + \left[\rho^*\text{Src}.\frac{N_x}{\rho^*}\right]^t + \left[\rho^*\text{Fall}.\frac{N_x}{\rho^*}\right]^t \tag{6.130}$$

The terms F_w^t and F_θ^t in the case of calculating gravity wave mode with small time step shown in (6.94) and (6.95) is able to be expressed as follows.

$$F_w^t = -\text{Adv}.w^t + \left[\overline{\rho^*\text{Buoy}.q}^\zeta + \overline{\rho^* f_c \overline{u}^x}^\zeta\right]^t + \left[G^{\frac{1}{2}}\text{Turb}.w + G^{\frac{1}{2}}\text{Diff}.w\right]^{t-\Delta t} \tag{6.131}$$

$$F_\theta^t = -\text{Adv}.\theta^t + \left[G^{\frac{1}{2}}\text{Turb}.\theta + G^{\frac{1}{2}}\text{Diff}.\theta\right]^{t-\Delta t} + [\rho^*\text{Src}.\theta]^t \tag{6.132}$$

6.3 Boundary condition

Finit difference of the basic equations shown in the previous section includes operators at $i-1$, $i+1$, $j-1$, ... For calculation of forecast variables at he grid points which locate along each boundary:

$$\begin{array}{llll} u; & i = 1, nx, & j = 1, ny - 1, & k = 1, nz - 1 \\ v; & i = 1, nx - 1, & j = 1, ny, & k = 1, nz - 1 \\ w; & i = 1, nx - 1, & j = 1, ny - 1, & k = 1, nz \\ \phi; & i = 1, nx - 1, & j = 1, ny - 1, & k = 1, nz - 1, \end{array} \tag{6.133}$$

where ϕ of the arbitrary scalar, and a boundary condition is needed to be given. We will explain about some kinds of a boundary condition in **CReSS** in the followings. In the fourth-order calculation, values at the grid points shifted one grid from the boundary are calculated with the second-order precision.

6.3.1 Lateral boundary condition

Four boundary conditions of periodic, fixed wall (mirror condition), non-gradient, and radiation, are applicable to a lateral boundary.

Periodic boundary condition

The periodic boundary condition is the condition that the values along the eastern (northern) boundary are equal to those along the western (southern) boundary during an integration.

In the cast that we use the periodic boundary condition in the east-west direction,

$$
\begin{aligned}
u_{1,j,k} &= u_{nx-2,j,k}, & u_{nx,j,k} &= u_{3,j,k} \\
v_{1,j,k} &= v_{nx-2,j,k}, & v_{nx-1,j,k} &= v_{2,j,k} \\
w_{1,j,k} &= w_{nx-2,j,k}, & w_{nx-1,j,k} &= w_{2,j,k} \\
\phi_{1,j,k} &= \phi_{nx-2,j,k}, & \phi_{nx-1,j,k} &= \phi_{2,j,k} \\
W_{1,j,k} &= W_{nx-2,j,k}, & W_{nx-1,j,k} &= W_{2,j,k},
\end{aligned}
\tag{6.134}
$$

or in the cast that we use the periodic boundary condition in the north-south direction,

$$
\begin{aligned}
u_{i,1,k} &= u_{i,ny-2,k}, & u_{i,ny-1,k} &= u_{i,2,k} \\
v_{i,1,k} &= v_{i,ny-2,k}, & v_{i,ny,k} &= v_{i,3,k} \\
w_{i,1,k} &= w_{i,ny-2,k}, & w_{i,ny-1,k} &= w_{i,2,k} \\
\phi_{i,1,k} &= \phi_{i,ny-2,k}, & \phi_{i,ny-1,k} &= \phi_{i,2,k} \\
W_{i,1,k} &= W_{i,ny-2,k}, & W_{i,ny-1,k} &= W_{i,2,k},
\end{aligned}
\tag{6.135}
$$

where ϕ is the arbitrary scalar.

Fixed wall boundary condition

In order to explain fixed wall boundary condition, for example, an east-and-west wall is shown here. Along the east and west boundaries, x differentiation in uncompressible equation of conservation of mass and free-slip condition are

$$
\partial\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial \zeta}\right) \Big/ \partial x = 0, \qquad \frac{\partial w}{\partial x} = 0.
\tag{6.136}
$$

It is drawn that the velocity u of x direction satisfys the following relation.

$$
\frac{\partial^2 u}{\partial x^2} = 0 \qquad \Longrightarrow \qquad u\left(x = -\Delta x, y, z\right) = u\left(x = \Delta x, y, z\right)
\tag{6.137}
$$

Under the situation, the fixed wall condition is determined as follows.

In the cast that we use the periodic boundary condition in the east-west direction,

$$
\begin{aligned}
u_{1,j,k} &= -u_{3,j,k}, & u_{nx,j,k} &= -u_{nx-2,j,k} \\
v_{1,j,k} &= v_{2,j,k}, & v_{nx-1,j,k} &= v_{nx-2,j,k} \\
w_{1,j,k} &= w_{2,j,k}, & w_{nx-1,j,k} &= w_{nx-2,j,k} \\
\phi_{1,j,k} &= \phi_{2,j,k}, & \phi_{nx-1,j,k} &= \phi_{nx-2,j,k} \\
W_{1,j,k} &= W_{2,j,k}, & W_{nx-1,j,k} &= W_{nx-2,j,k},
\end{aligned}
\tag{6.138}
$$

or in the cast that we use the periodic boundary condition in the north-south direction,

$$
\begin{aligned}
u_{i,1,k} &= u_{i,2,k}, & u_{i,ny-1,k} &= u_{i,nx-2,k} \\
v_{i,1,k} &= -v_{i,3,k}, & v_{i,ny,k} &= -v_{i,nx-2,k} \\
w_{i,1,k} &= w_{i,2,k}, & w_{i,ny-1,k} &= w_{i,nx-2,k} \\
\phi_{i,1,k} &= \phi_{i,2,k}, & \phi_{i,ny-1,k} &= \phi_{i,nx-2,k} \\
W_{i,1,k} &= W_{i,2,k}, & W_{i,ny-1,k} &= W_{i,nx-2,k},
\end{aligned}
\tag{6.139}
$$

where ϕ is the arbitrary scalar.

Non-gradient boundary condition

Non-gradient boundary condition is the condition that the gradient of forecast variables across boundary plane may be set to $grad\,(\phi) = 0$.

In the cast that we use the periodic boundary condition in the east-west direction,

$$
\begin{aligned}
u_{1,j,k} &= u_{3,j,k}, & u_{nx,j,k} &= u_{nx-2,j,k} \\
v_{1,j,k} &= v_{2,j,k}, & v_{nx-1,j,k} &= v_{nx-2,j,k} \\
w_{1,j,k} &= w_{2,j,k}, & w_{nx-1,j,k} &= w_{nx-2,j,k} \\
\phi_{1,j,k} &= \phi_{2,j,k}, & \phi_{nx-1,j,k} &= \phi_{nx-2,j,k} \\
W_{1,j,k} &= W_{2,j,k}, & W_{nx-1,j,k} &= W_{nx-2,j,k},
\end{aligned}
\tag{6.140}
$$

or in the cast that we use the periodic boundary condition in the north-south direction,

$$
\begin{aligned}
u_{i,1,k} &= u_{i,2,k}, & u_{i,ny-1,k} &= u_{i,nx-2,k} \\
v_{i,1,k} &= v_{i,3,k}, & v_{i,ny,k} &= v_{i,nx-2,k} \\
w_{i,1,k} &= w_{i,2,k}, & w_{i,ny-1,k} &= w_{i,nx-2,k} \\
\phi_{i,1,k} &= \phi_{i,2,k}, & \phi_{i,ny-1,k} &= \phi_{i,nx-2,k} \\
W_{i,1,k} &= W_{i,2,k}, & W_{i,ny-1,k} &= W_{i,nx-2,k},
\end{aligned}
\tag{6.141}
$$

where ϕ is the arbitrary scalar.

Radiational boundary condition

Radiational boundary condition is set up so that the radiational conditions with one-dimensional linear wave equation

$$
\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0
\tag{6.142}
$$

is satisfied for stopping the action of the unnatural solution produced in a boundary plane. Adding the attenuation term,

$$\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = -\gamma \phi \tag{6.143}$$

is used here. As nesting from a model output for a larger domain, such as objective analysis data, as an initial and a boundary values. The difference between **CReSS** and the model for a larger domain for each physical amount is used for a boundary condition.

$$\frac{\partial (\phi_f - \phi_c)}{\partial t} + c \frac{\partial (\phi_f - \phi_c)}{\partial x} = -\gamma (\phi_f - \phi_c) \tag{6.144}$$

where ϕ is the arbitrary physical quantities, bottom characters f and c indicate the values of the larger model and this model, respectively.

There are various methods in the way to solve phase velocity c in the above radiational condition. The three methods can be chosen in **CReSS**. ϕ and $\phi_f - \phi_c$ is expressed with Φ below.

[(1) Method to calculate phase velocity for each grid point](1) Method to calculate phase velocity for each

grid point
    In this case, equations excluded the attenuation term on the right side from the equations (6.143), (6.144) is solved about phase velocity c. Non-dimensional phase velocity is directly solved.

$$C_b^t = c_b^t \frac{\Delta t}{\Delta x} = -\frac{\Phi_b^{t+\Delta t} - \Phi_b^{t-\Delta t}}{2\Phi_{b-1}^t - \Phi_b^{t+\Delta t} - \Phi_b^{t-\Delta t}}, \qquad -1 \le C_b^t \le 0, \quad \text{western edge or southern edge} \tag{6.145}$$

$$C_b^t = c_b^t \frac{\Delta t}{\Delta x} = -\frac{\Phi_b^{t+\Delta t} - \Phi_b^{t-\Delta t}}{\Phi_b^{t+\Delta t} + \Phi_b^{t-\Delta t} - 2\Phi_{b-1}^t}, \qquad 0 \le C_b^t \le 1, \quad \text{eastern edge or northern edge} \tag{6.146}$$

These phase velocities should be calculated for each forecast variables. In the actual calculation, the following values are used.

$$C_b^t = c_b^t \frac{\Delta t}{\Delta x} = -\frac{\Phi_{b-1}^t - \Phi_{b-1}^{t-2\Delta t}}{2\Phi_{b-2}^{t-\Delta t} - \Phi_{b-1}^t - \Phi_{b-1}^{t-2\Delta t}}, \qquad -1 \le C_b^t \le 0, \, westernmost or southernmost \tag{6.147}$$

$$C_b^t = c_b^t \frac{\Delta t}{\Delta x} = -\frac{\Phi_{b-1}^t - \Phi_{b-1}^{t-2\Delta t}}{\Phi_{b-1}^t + \Phi_{b-1}^{t-2\Delta t} - 2\Phi_{b-2}^{t-\Delta t}}, \qquad 0 \le C_b^t \le 1, \, easternmost or northernmost \tag{6.148}$$

where bottom character b is the number of grid point on the boundary given by the equation (6.133).

[(2) Method to calculate constant phase velocity in vertical direction](2) Method to calculate constant

phase velocity in vertical direction
    Wave equation which assumed constant phase velocity in vertical is considered as follows.

$$\int_{z_{sfc}}^{z_{top}} \frac{\partial \Phi}{\partial t} dz + c \int_{z_{sfc}}^{z_{top}} \frac{\partial \Phi}{\partial x} dz = -\gamma \Phi \tag{6.149}$$

Using the value calculated at the inner grid point on preceding time step,

$$C_b^t = c_b^t \frac{\Delta t}{\Delta x} = -\frac{\sum_{k=kmin}^{kmax} \left\{ \left(\Phi_{b-1}^t - \Phi_{b-1}^{t-2\Delta t} \right) sgn \left(2\Phi_{b-2}^{t-\Delta t} - \Phi_{b-1}^t - \Phi_{b-1}^{t-2\Delta t} \right) \right\}}{\sum_{k=kmin}^{kmax} \left| 2\Phi_{b-2}^{t-\Delta t} - \Phi_{b-1}^t - \Phi_{b-1}^{t-2\Delta t} \right|},$$

$$\cdots \quad -1 \leq C_b^t \leq 0, \quad \text{western edge or southern edge} \tag{6.150}$$

$$C_b^t = c_b^t \frac{\Delta t}{\Delta x} = -\frac{\sum_{k=kmin}^{kmax} \left\{ \left(\Phi_{b-1}^t - \Phi_{b-1}^{t-2\Delta t} \right) sgn \left(\Phi_{b-1}^t + \Phi_{b-1}^{t-2\Delta t} - 2\Phi_{b-2}^{t-\Delta t} \right) \right\}}{\sum_{k=kmin}^{kmax} \left| \Phi_{b-1}^t + \Phi_{b-1}^{t-2\Delta t} - 2\Phi_{b-2}^{t-\Delta t} \right|},$$

$$\cdots \quad 0 \leq C_b^t \leq 1, \quad \text{eastern edge or northern edge} \tag{6.151}$$

where $kmin$ and $kmax$ are

$$u, \ v, \ \phi \ ; \quad kmin = 2, \ kmax = nz - 2 \tag{6.152}$$
$$w \ ; \quad kmin = 2, \ kmax = nz - 1. \tag{6.153}$$

[(3) Method to add advectional velocity to constant phase velocity](3) Method to add advectional velocity

to constant phase velocity

Phase velocities should be calculated for each forecast variable. c^* is given as the propagation velocity of typical gravity wave, such as about 30 m s^{-1}. Using x-component velocity u for the eastern and western boundaries and y-component velocity v for the northern and southern boundaries, a boundary condition is given as follows.

$$C_b^t = c_b^t \frac{\Delta t}{\Delta x} = (u - c^*) \frac{\Delta t}{\Delta x}, \qquad -1 \leq C_b^t \leq 0, \ \text{westernmost} \tag{6.154}$$

$$C_b^t = c_b^t \frac{\Delta t}{\Delta x} = (u + c^*) \frac{\Delta t}{\Delta x}, \qquad 0 \leq C_b^t \leq 1, \ \text{easternmost} \tag{6.155}$$

$$C_b^t = c_b^t \frac{\Delta t}{\Delta y} = (v - c^*) \frac{\Delta t}{\Delta y}, \qquad -1 \leq C_b^t \leq 0, \ \text{southernmost} \tag{6.156}$$

$$C_b^t = c_b^t \frac{\Delta t}{\Delta y} = (v + c^*) \frac{\Delta t}{\Delta y}, \qquad 0 \leq C_b^t \leq 1, \ \text{northernmost} \tag{6.157}$$

[(4) Method to use constant phase velocity](4) Method to use constant phase velocity

Only c^* in (3) is used on assumption that a wave always goes away out of the calculation domain. Therefore, phase velocities at all grid points on the lateral boundary plane are the same value. However, a sign becomes reverse between eastern and western sides, and southern and northern sides.

In the case of (3) and (4), the same phase velocity as (2) is used for a vertical velocity component, such as u on the eastern and western planes and v on the southern and northern planes.

By using the phase velocity solved in (1), (2), (3), and (4), the value at a grid point on the boundary at next time step is,

for the variable integrated with small time step $\Delta \tau$,

$$\Phi_b^{\tau+\Delta\tau} = \Phi_b^{\tau} - \frac{C_b^t}{1-C_b^t}\left(\Phi_{b-1}^t - \Phi_b^{t-\Delta t}\right)\frac{\Delta\tau}{\Delta t} - \gamma\Phi_b^{t-\Delta t}\Delta\tau, \qquad western\ edge\ or\ southern\ edge \tag{6.158}$$

$$\Phi_b^{\tau+\Delta\tau} = \Phi_b^{\tau} + \frac{C_b^t}{1+C_b^t}\left(\Phi_{b-1}^t - \Phi_b^{t-\Delta t}\right)\frac{\Delta\tau}{\Delta t} - \gamma\Phi_b^{t-\Delta t}\Delta\tau, \qquad eastern\ edge\ or\ northern\ edge \tag{6.159}$$

for the variable integrated with large time step Δt,

$$\Phi_b^{t+\Delta t} = \Phi_b^{t-\Delta t} - \frac{2C_b^t}{1-C_b^t}\left(\Phi_{b-1}^t - \Phi_b^{t-\Delta t}\right) - \gamma\Phi_b^{t-\Delta t}\Delta t, \qquad westernmost\ or\ southernmost \tag{6.160}$$

$$\Phi_b^{t+\Delta t} = \Phi_b^{t-\Delta t} + \frac{2C_b^t}{1+C_b^t}\left(\Phi_{b-1}^t - \Phi_b^{t-\Delta t}\right) - \gamma\Phi_b^{t-\Delta t}\Delta t, \qquad easternmost\ or\ northernmost \tag{6.161}$$

where attenuation constant γ has the dimension of time inverse number. It is thought that the inverse number $1/\gamma$ is suitable more than one hour and less than twelve hours.

6.3.2   Boundary conditions at the top and bottom

Two kinds of a boundary condition, fixed wall (mirror condition) and non-gradient, are applicable to boundary conditions at the top and bottom.

Fixed wall boundary condition

Basically, it can be considered as same as lateral fixed wall boundary condition. However, when we use the terrain-following coordinate system, the relation between the velocity on Cartesian coordinate system and contravariant velocity should be considered as follows.

$$W = (uJ_{31} + vJ_{32} + w)\Big/G^{\frac{1}{2}} \tag{6.162}$$

Note that vertical velocity W when velocity is 0 on the boundary plane.

On the boundary plane at the top $(k = nz - 1)$,

$$W = 0, \qquad w = 0 \tag{6.163}$$

and on the boundary plane at the bottom $(k = 2)$,

$$W = 0, \qquad w = -uJ_{31} - vJ32. \tag{6.164}$$

Therefore, boundary condition is given as follows.

$$
\begin{aligned}
u_{i,j,1} &= u_{i,j,2}, & u_{i,j,nz-1} &= u_{i,j,nz-2} \\
v_{i,j,1} &= v_{i,j,2}, & v_{i,j,nz-1} &= v_{i,j,nz-2} \\
\phi_{i,j,1} &= \phi_{i,j,2}, & \phi_{i,j,nz-1} &= \phi_{i,j,nz-2} \\
W_{i,j,2} &= 0, \quad W_{i,j,1} = -W_{i,j,3}, & W_{i,j,nz-1} &= 0, \quad W_{i,j,nz} = -W_{i,j,nz-2}
\end{aligned}
\tag{6.165}
$$

where ϕ is the arbitrary scalar. By replacing vertical velocity W at the top and bottom boundaries with the equation (6.162),
on boundary plane at the top $(k = nz - 1)$,

$$w_{i,j,nz-1} = 0 \tag{6.166}$$

$$w_{i,j,nz} = -\left(\overline{\overline{u^\zeta J_{31}}}^x + \overline{\overline{v^\zeta J_{32}}}^y + W\overline{G^{\frac{1}{2}}}^\zeta \right)_{i,j,nz-2} \tag{6.167}$$

on the boundary plane at the bottom $(k = 2)$,

$$w_{i,j,2} = -\left(\overline{\overline{u^\zeta J_{31}}}^x + \overline{\overline{v^\zeta J_{32}}}^y \right)_{i,j,2} \tag{6.168}$$

$$w_{i,j,1} = -\left(\overline{\overline{u^\zeta J_{31}}}^x + \overline{\overline{v^\zeta J_{32}}}^y + W\overline{G^{\frac{1}{2}}}^\zeta \right)_{i,j,3} \tag{6.169}$$

where pressure perturbation p' on the boundary plane at the bottom is given by extrapolation as follows.

$$p'_{i,j,1} = 2p'_{i,j,2} - p'_{i,j,3} \tag{6.170}$$

Non-gradient boundary condition

Similarly to the lateral boundary,

$$
\begin{array}{ll}
u_{i,j,1} = u_{i,j,2}, & u_{i,j,nz-1} = u_{i,j,nz-2} \\
v_{i,j,1} = v_{i,j,2}, & v_{i,j,nz-1} = v_{i,j,nz-2} \\
w_{i,j,1} = w_{i,j,2}, & w_{i,j,nz} = w_{i,j,nz-1} \\
\phi_{i,j,1} = \phi_{i,j,2}, & \phi_{i,j,nz-1} = \phi_{i,j,nz-2} \\
W_{i,j,1} = W_{i,j,2}, & W_{i,j,nz} = W_{i,j,nz-1}
\end{array}
\tag{6.171}
$$

where ϕ is the arbitrary scalar.

6.3.3   Sponge layer

In **CReSS**, the sponge layer set as arbitrary thickness is able to be set up. This is a layer for suppressing and absorbing wave reflections near the boundary plane. An additional term is added to the basic equation as follows.

$$
\frac{\partial \left(\rho^* \phi\right)}{\partial t} = -\text{Adv.}\phi + ... - \gamma_h \left(x, y\right) \rho^* \left(\phi - \phi_e\right) - \gamma_v \left(z\right) \rho^* \left(\phi - \phi_e\right)
\tag{6.172}
$$

where ϕ is an arbitrary forecast variable, ϕ_e is a variable of external data, such as an objective analysis, $\gamma_h \left(x, y\right)$ is an attenuation coefficient which becomes small toward each horizontal boundary plane. $\gamma_v \left(z\right)$ is attenuation coefficient which becomes small toward each vertical boundary plane.

For example, in the case of the west lateral boundary layer which made thickness of sponge layer d in horizontal, $\gamma_h \left(x, y\right)$ is given as follows.

$$
\gamma_h = \begin{cases}
\alpha_h \left(1 - \dfrac{x}{d}\right)^3, & x \le d \\
\\
0, & x > d
\end{cases}
\tag{6.173}
$$

In the case of the bottom boundary layer which made minimum height of sponge layer in vertical, $\gamma_v \left(z\right)$ is given as follows.

$$
\gamma_v = \begin{cases}
0, & z < z_{low} \\
\\
\alpha_v \left\{1 - cos \left(\pi \dfrac{z - z_{low}}{z_{top} - z_{low}}\right)\right\}, & z \ge z_{low}
\end{cases}
\tag{6.174}
$$

where α_h and α_v are horizontal and vertical attenuation constants, and have dimension of the inverse number of time. These inverse numbers $1/\alpha_h$ and $1/\alpha_v$ is called e-folding time. It is appropriate that these constants are set up in the range about $1/100 \sim 1/300$ s^{-1} . In addition, it is suitable for the thickness of sponge layer to set up in the number of several grid points in horizontal and the thickness of vertical layer about the upper surface to $1/3$ in vertical.

This term is calculated at $t - \Delta t$. About the forecast variable calculated at a small time step, it is treated as a value of a big time step $2\Delta t$ like advection term and numerical viscosity term.

The adjusted value to adjust was external data's value ϕ_e, such as objective analysis, in the equation (6.172). However, when a model is performed independently, and there is no objective analysis data for a forecast variable, it can adjust to initial value ϕ_0 or basic state value $\bar{\phi}$ as follows.

$$\frac{\partial \left(\rho^* \phi\right)}{\partial t} = -\text{Adv.}\phi + ... - \gamma_h \left(x, y\right) \rho^* \left(\phi - \phi_0\right) - \gamma_v \left(z\right) \rho^* \left(\phi - \phi_0\right) \tag{6.175}$$

$$\frac{\partial \left(\rho^* \phi\right)}{\partial t} = -\text{Adv.}\phi + ... - \gamma_h \left(x, y\right) \rho^* \left(\phi - \bar{\phi}\right) - \gamma_v \left(z\right) \rho^* \left(\phi - \bar{\phi}\right) \tag{6.176}$$