Chapter 4

Physical Processes of Cloud and Precipitation

Cloud physics is roughly divided into cloud microphysics and cloud dynamics. Because they are closely concerned in each other, knowledges of cloud physics is necessary for understanding cloud dynamics.

Ascending a humid air parcel in atmosphere, clouds generate in association with a conversion from water vapor to cloud and precipitation particles and precipitation is induced. We generally classify the formation processes of precipitation into "warm rain" and "cold rain (ice-phase rain)." The "warm rain" is rainfall which has no ice-phase process, and clouds form below the altitude of 0°C. Such clouds is called "warm clouds." The "cold rain" is rainfall of which ice-phase processes considerably contribute for the growth of precipitation particles. Such clouds is called "cold clouds." The clouds form not only below the altitude of 0°C but also above the altitude. The clouds have both a liquid phase and an ice-phase. There are the following categories in modeling of these clouds.

- Parametarization of "warm rain" with a bulk method
- Parametarization containing an ice-phase with a bulk method
- A method predicting time development of size distribution with dividing the sizes of liquid-phase particles into several bins (a bin method)
- A bin method as same as the above but containing an ice-phase
- A bin method containing categorized aerosols
- A method of which a bin method is used for a liquid phase and a bulk method for an ice-phase (A hybrid type)

Bulk method parametarizations of "warm rain" and "cold rain" are used in the ***CReSS***. Hereafter, modelings of these two methods are described.

4.1 Parametarization of "Warm Rain" with a Bulk Method

4.1.1 Equations for "Warm Rain" Process of Cloud and Precipitation

The parametarization of "warm rain" with a bulk method has the following three categories of water substance[1].

Sign	Meaning	Contents
q_v	Mixing ratio of water vapor	Water which exists in the atmosphere in the state of gas
q_c	Mixing ratio of cloud water	Its fall velocity can be regarded as zero. It corresponds to cloud drops in the actual atmosphere. The cloud drop is ordinary a small particle of which its diameter is less than 100 μm.
q_r	Mixing ratio of rain water	It corresponds to raindrops, which have a meaningful fall velocity, in the actual atmosphere.

Modeling variables involved in the "warm rain" process are three variables for water substance [kg kg^{-1}] and potential temperature (temperature) [K]. Using equations shown in section 2.1(we can explain by using the equations in which orography is not considered), the equations for the "warm rain" process are simply written as follows.

$$\frac{\partial \bar{\rho}\theta}{\partial t} = \text{Adv.}\theta + \text{Turb.}\theta - \bar{\rho}w\frac{\partial \bar{\theta}}{\partial z} + \frac{\bar{\rho}\mathcal{L}_v}{C_p\Pi}\left(CN_{vc} - EV_{cv} - EV_{rv}\right) \tag{4.1}$$

$$\frac{\partial \bar{\rho}q_v}{\partial t} = \text{Adv.}q_v + \text{Turb.}q_v - \bar{\rho}\left(CN_{vc} - EV_{cv} - EV_{rv}\right) \tag{4.2}$$

$$\frac{\partial \bar{\rho}q_c}{\partial t} = \text{Adv.}q_c + \text{Turb.}q_c + \bar{\rho}\left(CN_{vc} - EV_{cv} - CN_{cr} - CL_{cr}\right) \tag{4.3}$$

$$\frac{\partial \bar{\rho}q_r}{\partial t} = \text{Adv.}q_r + \text{Turb.}q_r + \bar{\rho}\left(CN_{cr} + CL_{cr} - EV_{rv}\right) + \frac{\partial}{\partial z}\left(\bar{\rho}U_r q_r\right) \tag{4.4}$$

where Adv.ϕ and Turb.ϕ represent the terms of advection and subgrid-scale turbulance. $\mathcal{L}_v$, C_p and Π are latent heat of evaporation of water [J kg^{-1}], specific heat at constant pressure in a dry air [J K kg^{-1}] and the Exner function, respectively. The last term of the right side of Eq. (4.4) represents a flux divegence of q_r in association with falling of rainwater. Microphisical processes considered here are as follows.

Sign	Contents
CN_{vc}	Conversion from water vapor to cloud water by condensation (condensation)
EV_{cv}	Conversion from cloud water to water vapor by evaporation (evaporation)
EV_{rv}	Conversion from rain water to water vapor by evaporation (evaporation)
CN_{cr}	Conversion from cloud water to rain water by coalescence growth. It corresponds to growing from a cloud droplet to the size of a raindrop by coalescence or diffusion of water vapor (autoconversion).
CL_{cr}	Conversion from cloud water to rain water by collision. A process of which a large waterdrop captures a small waterdrop (collection).

All of these amounts are defined as a positive value. A process of condensation from water vapor to rain

[1]Rain water which has a diameter of 0.1-0.5 mm is sometimes called drizzle. Here, drizzle is contained in a category of rain.

water is neglected. Calculations of those processes are shown in the next section.

4.1.2 Microphysical Processes

Conversion between Water Vapor and Cloud Water: $-CN_{vc} + EV_{cv}$

As the same as Klemp and Wilhelmson (1978), we use the method of the moist saturation adjustment by Soong and Ogura (1973). This method will be explained in Section 4.2.5.

Saturation Mixing Ratio (Tetens Equation): q_{vsw}

Saturation mixing ratio over water q_{vsw} is represented by using the Tetens equation.

$$q_{vsw} = \epsilon \frac{610.78}{p} \exp\left(17.269 \frac{\Pi\theta - 273.16}{\Pi\theta - 35.86}\right) \tag{4.5}$$

Here, ϵ is the ratio with the molecular weight of water vapor and the molecular weight of dry air.

Conversion from Cloud Water to the Rain Water: CN_{cr}, CL_{cr}

The conversion from cloud water to rain water with coalescence growth (CN_{cr}) and conversion from cloud water to rain water (CL_{cr}) is calculated by using the Kessler (1969)'s parameterization.

$$CN_{cr} = k_1 \left(q_c - a\right) \tag{4.6}$$

$$CL_{cr} = k_2 q_c q_r^{0.875} \tag{4.7}$$

where

$$k_1 = 0.001 \quad [\mathrm{s}^{-1}] \tag{4.8}$$

$$a = 0.001 \quad [\mathrm{kg\ kg}^{-1}] \tag{4.9}$$

$$k_2 = 2.2 \quad [\mathrm{s}^{-1}]. \tag{4.10}$$

Evaporation of Rain Water: EV_{rv}

In the same way as Ogura and Takahashi (1971) and Klemp and Wilhelmson (1978), evaporation of rain water is represented as a following.

$$EV_{rv} = \frac{1}{\bar{\rho}} \frac{\left(1 - q_v / q_{vsw}\right) C \left(\bar{\rho} q_r\right)^{0.525}}{5.4 \times 10^5 + 2.55 \times 10^6 / \left(p q_{vsw}\right)} \tag{4.11}$$

where C is a ventilation factor given by

$$C = 1.6 + 124.9 \left(\bar{\rho} q_r\right)^{0.2046}. \tag{4.12}$$

Fall Velocity of Rain: U_r

Adding change of density to the equation of Soong and Ogura (1973), fall velocity U_r of the last term in the right side of the equation (4.4) is given by

$$U_r = 36.34\,(\bar{\rho}q_r)^{0.1346}\left(\frac{\rho_0}{\bar{\rho}}\right). \tag{4.13}$$

where ρ_0 is density [kg m^{-3}] at the surface in the basic state and the unit of U_r is [m s^{-1}]. Precipitation at the surface is calculated by using this fall velocity. Note that a difference in z^* cordinate system (ζ cordinate system) is multiplied by a matrix, as shown in equations (2.58).

4.2 Parameterization of Cloud and Precipitation with the Mixed-phase Processes

4.2.1 Parameterization of the Mixed-phase Processes with a Bulk Method

We formulate the parameterization of the cloud and precipitation processes containing ice-phase with a bulk method which are used in a cloud model. Here, conversions of water substance and changes of temperature and mixing ratio of water vapor by the conversions are considered. When we use a bulk method, the water in the atmosphere is divided into some categories (e.g. rain, snow and graupel) The categories are formulized with the typical variables (usually, mixing ratio or mixing ratio and number concentration), and their time developments are calculated. Therefore, when we use a bulk method, the definition of each variable must be clear. The way of categorization and variables are different in each model.

The following physical processes are considered in formulization of parameterization containing an ice-phase. In the formulization, the type of a particle must be considered.

- Primary and secondary nucleation of ice crystals
- Growth and decline of the particle with water vapor diffusion.
- Growth with collision between particles
- Breakup of a particle (breakup of a raindrop)
- Conversion to another category (Conversion from the cloud water to rain water, and from cloud water to snow, and from snow to graupel. etc.)
- Freezing and melting
- Shedding of un-freezing water
- Gravity falling

Although the definitions of the cloud physical valiables and its treatment are different in each model, we show two types of models containing an ice-phase in the following.

- A model which calculates only the time development equation of the mixing ratio of each category
- A model which calculates the graupel, cloud ice, snow, and their number concentrations adding the above equation

Hereafter, we summarize the formulizations by Murakami (1999), Murakami et al. (1994) and Murakami (1990).

We consider the six categories for the physical processes of cloud and precipitation: water vapor, cloud water, rain water, cloud ice, snow and graupel. In addition to the six categories, there is a model which makes fog water and hail another categories.

Sign	Meaning	Contents
θ	Potential temperture	$\theta = \bar{\theta} + \theta'$
q_v	Mixing ratio of water vapor	Water which exists in the atmosphere in the state of gas.
q_c	Mixing ratio of cloud water	Liquid water of which a particle diameter is small and fall velocity is negligible. It can move with atmospheric motion.
q_r	Mixing ratio of rain water	Usually, the particle of a liquid water with a diameter more than 100 μm is called rain. It is expressed as "rain water" in a model. Fall velocity depending on the size of a particle is meaningful. Although it moves in association with the atmospheric motion in horizontal, it falls and drops out of an air mass in vertical.
q_i	Mixing ratio of cloud ice	A minute crystal of ice which is called "ice crystal" in cloud physics. Usually, a diameter is less than 100 μm . Fall velocity is so small that it is negligible.
q_s	Mixing ratio of snow	In a model, it means a solid precipitation particle with density of about 0.1 g cm^{-3} and fall velocity of about 1 m s^{-1} . It corresponds to a snow crystal or snow flakes, etc.
q_g	Mixing ratio of graupel	In a model, it means a solid precipitation particle with density of about 0.4 g cm^{-3} and fall velocity of about 1∼4 m s^{-1} . It corresponds to a snow crystal with cloud droplets, snow flakes with cloud droplets, graupel, etc.
q_h	Mixing ratio of hail	In a model, it means a solid precipitation particle with density of about 0.9 g cm^{-3} and fall velocity of about 10 m s^{-1} fall speed.In an actual cloud particle, frozen rain, hail, etc. correspond. hail is contained in graupel in CReSS.
N_i	number concentration of cloud ice	Expressing cloud ice in a model, number concentration is sometimes regarded as a variable.
N_s	number concentration of snow	The same in case of snow.
N_g	number concentration of graupel	The same in case of graupel.

Here, the units of potential temperature, mixing ratio and number concentration are [K], [kg kg^{-1}] and [m^{-3}], respectively. We often use [g kg^{-1}] as the unit of mixing ratio.

4.2.2 The Equation System for Cloud and Precipitation Processes

The time development equations of potential temperature, water vapor and particles of cloud and precipitation are used here. Considering number concentrations of particles, the time development equations of number concentration of cloud ice, snow, and graupel are needed. These describe briefly the equation system shown in Section 2.1.

The time development equation of a potential temperature and a mixing ratio of water is expressed as

$$\frac{\partial \bar{\rho}\theta}{\partial t} = \mathrm{Adv}.\theta + \mathrm{Turb}.\theta - \bar{\rho}w\frac{\partial \bar{\theta}}{\partial z} + \bar{\rho}\left(\mathrm{Src}.\theta_V + \mathrm{Src}.\theta_S + \mathrm{Src}.\theta_F\right) \tag{4.14}$$

$$\frac{\partial \bar{\rho}q_v}{\partial t} = \mathrm{Adv}.q_v + \mathrm{Turb}.q_v + \bar{\rho}\mathrm{Src}.q_v \tag{4.15}$$

$$\frac{\partial \bar{\rho}q_c}{\partial t} = \mathrm{Adv}.q_c + \mathrm{Turb}.q_c + \bar{\rho}\mathrm{Src}.q_c + \bar{\rho}\mathrm{Fall}.q_c \tag{4.16}$$

$$\frac{\partial \bar{\rho}q_r}{\partial t} = \mathrm{Adv}.q_r + \mathrm{Turb}.q_r + \bar{\rho}\mathrm{Src}.q_r + \bar{\rho}\mathrm{Fall}.q_r \tag{4.17}$$

$$\frac{\partial \bar{\rho}q_i}{\partial t} = \mathrm{Adv}.q_i + \mathrm{Turb}.q_i + \bar{\rho}\mathrm{Src}.q_i + \bar{\rho}\mathrm{Fall}.q_i \tag{4.18}$$

$$\frac{\partial \bar{\rho}q_s}{\partial t} = \mathrm{Adv}.q_s + \mathrm{Turb}.q_s + \bar{\rho}\mathrm{Src}.q_s + \bar{\rho}\mathrm{Fall}.q_s \tag{4.19}$$

$$\frac{\partial \bar{\rho}q_g}{\partial t} = \mathrm{Adv}.q_g + \mathrm{Turb}.q_g + \bar{\rho}\mathrm{Src}.q_g + \bar{\rho}\mathrm{Fall}.q_g \tag{4.20}$$

where characters of v, c, r, i, s, and g, which are attached at the bottom, express cloud water, rain water, cloud ice, snow, and graupel, respectively. Hereafter, we sometimes use x or y as representative characters.

The meaning of each term is shown below.

Adv.ϕ	Advection term of potential temperature or mixing ratio of water substance
Turb.ϕ	Diffusion term of potential temperature or mixing ratio of water substance by a subgrid-scale turbulance
Src.θ_V	Source term of potential temperature in association with condensation and evaporation
Src.θ_S	Source term of potential temperature in association with sublimation
Src.θ_F	Source term of potential temperature in association with freezeing and melting
Src.q_x	Source term of the mixing ratio of water
Fall.q_x	Term of falling water substance (precipitation)

The time development equations of number concentrations of cloud ice, snow, and graupel are

$$\frac{\partial N_i}{\partial t} = \mathrm{Adv}.\frac{N_i}{\bar{\rho}} + \mathrm{Turb}.\frac{N_i}{\bar{\rho}} + \bar{\rho}\mathrm{Src}.\frac{N_i}{\bar{\rho}} + \bar{\rho}\mathrm{Fall}.\frac{N_i}{\bar{\rho}} \tag{4.21}$$

$$\frac{\partial N_s}{\partial t} = \mathrm{Adv}.\frac{N_s}{\bar{\rho}} + \mathrm{Turb}.\frac{N_s}{\bar{\rho}} + \bar{\rho}\mathrm{Src}.\frac{N_s}{\bar{\rho}} + \bar{\rho}\mathrm{Fall}.\frac{N_s}{\bar{\rho}} \tag{4.22}$$

$$\frac{\partial N_g}{\partial t} = \mathrm{Adv}.\frac{N_g}{\bar{\rho}} + \mathrm{Turb}.\frac{N_g}{\bar{\rho}} + \bar{\rho}\mathrm{Src}.\frac{N_g}{\bar{\rho}} + \bar{\rho}\mathrm{Fall}.\frac{N_g}{\bar{\rho}} \tag{4.23}$$

. where characters of i, s, g, which are attached at the bottom, express water vapor, cloud ice, snow, and graupel, respectively. Hereafter, we sometimes use x or y as representative characters.

The meaning of each term is shown below.

Adv. $N_x/\bar{\rho}$	Advection term of number concentration of water in the solid state
Turb. $N_x/\bar{\rho}$	Diffusion term of number concentration of water in the solid state by a subgrid-scale turbulence
Src. $N_x/\bar{\rho}$	Source term of the number concentration of water in the solid state
Fall. $N_x/\bar{\rho}$	Term of change of number concentration of water in the solid state in association with precipitation

Source terms of those equations are as follows.

Source term of the equation (4.14) of potential temperature θ: $\mathrm{Src.}\theta_V + \mathrm{Src.}\theta_S + \mathrm{Src.}\theta_F$

$$\mathrm{Src.}\theta_V = \frac{\mathcal{L}_v}{C_p \Pi} VD_{vr} \tag{4.24}$$

$$\mathrm{Src.}\theta_S = \frac{\mathcal{L}_s}{C_p \Pi} \left(NUA_{vi} + VD_{vi} + VD_{vs} + VD_{vg} \right) \tag{4.25}$$

$$\begin{aligned} \mathrm{Src.}\theta_F = \frac{\mathcal{L}_f}{C_p \Pi} & (NUF_{ci} + NUC_{ci} + NUH_{ci} + CL_{cs} + CL_{cg} + CL_{ri} + CL_{rs} + CL_{rg} \\ & - ML_{ic} - ML_{sr} - ML_{gr} + FR_{rg} - SH_{sr} - SH_{gr}) \end{aligned} \tag{4.26}$$

Source term of the equation for mixing ratio of water vapor q_v (4.15):

$$\mathrm{Src.}q_v = -NUA_{vi} - VD_{vr} - VD_{vi} - VD_{vs} - VD_{vg} \tag{4.27}$$

Source term of the equation for mixing ratio of cloud water q_c (4.16):

$$\mathrm{Src.}q_c = -NUF_{ci} - NUC_{ci} - NUH_{ci} - CL_{cr} - CL_{cs} - CL_{cg} - CN_{cr} + ML_{ic} \tag{4.28}$$

Source term of the equation for mixing ratio of rain water q_r (4.17):

$$\begin{aligned} \mathrm{Src.}q_r = & VD_{vr} + CL_{cr} - CL_{ri} - CL_{rs} - CL_{rg} + CN_{cr} \\ & + ML_{sr} + ML_{gr} - FR_{rg} + SH_{sr} + SH_{gr} \end{aligned} \tag{4.29}$$

Source term of the equation for mixing ratio of cloud ice q_i (4.18):

$$\begin{aligned} \mathrm{Src.}q_i = & NUA_{vi} + NUF_{ci} + NUC_{ci} + NUH_{ci} \\ & + VD_{vi} - CL_{ir} - CL_{is} - CL_{ig} - CN_{is} - ML_{ic} + SP_{si} + SP_{gi} \end{aligned} \tag{4.30}$$

Source term of the equation for mixing ratio of snow q_s (4.19):

$$\begin{aligned} \mathrm{Src.}q_s = & -SP_{si} + VD_{vs} + CL_{cs} + CL_{rs}\alpha_{rs} + CL_{is} - CL_{sr}\left(1 - \alpha_{rs}\right) - CL_{sg} \\ & + CN_{is} - CN_{sg} - ML_{sr} - SH_{sr} \end{aligned} \tag{4.31}$$

Source term of the equation for mixing ratio of graupel q_g (4.20):

$$\begin{aligned} \mathrm{Src.}q_g = & -SP_{gi} + VD_{vg} + PG_g + CL_{ri} + CL_{ir} + \left(CL_{rs} + CL_{sr}\right)\left(1 - \alpha_{rs}\right) \\ & + CN_{sg} - ML_{gr} + FR_{rg} - SH_{gr} \end{aligned} \tag{4.32}$$

Source term of the equation for number concentration of cloud ice $\frac{N_i}{\bar{\rho}}$ (4.21):

$$\begin{aligned} \mathrm{Src.}\frac{N_i}{\bar{\rho}} = & \frac{1}{m_{i0}} NUA_{vi} + \frac{N_c}{\bar{\rho} q_c} \left(NUF_{ci} + NUC_{ci} + NUH_{ci} \right) + SP_{si}^N + SP_{gi}^N \\ & + \frac{N_i}{\bar{\rho} q_i} \left(VD_{vi} - CL_{ir} - CL_{is} - CL_{ig} - ML_{ic} \right) - AG_i^N - \frac{1}{m_{s0}} CN_{is} \end{aligned} \tag{4.33}$$

Source term of the equation for number concentration of snow $\frac{N_s}{\bar{\rho}}$ (4.22):

$$\mathrm{Src.}\frac{N_s}{\bar{\rho}} = \frac{N_s}{\bar{\rho} q_s} \left(VD_{vs} - ML_{sr} \right) - CL_{sr}^N \left(1 - \alpha_{rs}\right) - CL_{sg}^N - AG_s^N + \frac{1}{m_{s0}} CN_{is} - CN_{sg}^N \tag{4.34}$$

Source term of the equation for number concentration of graupel $\frac{N_g}{\bar{\rho}}$ (4.23):

$$\text{Src.}\frac{N_g}{\bar{\rho}} = \frac{N_g}{\bar{\rho} q_g} \left(VD_{vg} - ML_{gr}\right) + CL_{ri}^N + CL_{rs}^N \left(1 - \alpha_{rs}\right) + CN_{sg}^N + FR_{rg}^N \tag{4.35}$$

were $\mathcal{L}_v, \mathcal{L}_s$, and $\mathcal{L}_f$ are latent heat with evaporation, sublimation, and melting [J kg^{-1}], respectively. C_p is specific heat at the constant pressure [J K kg^{-1}] in dry air. Π is the Exner function. m_{i0} and m_{s0} are the minimum masses of cloud ice and snow [kg], respectively. The next table shows the meaning of each term in the above equations. Figure 4.1 shows the correlation between categories. In Section 4.2.4 conversion terms, which consist of these source terms, are formulated.

Sign	Contents
NUA_{vi}	Deposition or sorption nucleation
NUF_{ci}	Condensation-freezing nucleation
NUC_{ci}	Contact-freezing nucleation
NUH_{ci}	Homogeneous-freezing nucleation
SP	Secondary nucleation of ice crystals
VD	Vapor deposition, evaporation and sublimation
CL	Collision collection
PG	Growth of graupel with collisional collection （graupel produciton）
AG	Aggregation
CN	Conversion from a certain category to the other categories （conversion）
ML	Melting
FR	Freezing
SH	Shedding of liquid water
SP^N	Secondary nucleation of ice crystals for number concentration
CL^N	Collisional collection for number concentration （collection）
AG^N	Aggregation for number concentration （aggregation）
CN^N	Conversion from a certain category to the other categories for number concentration （conversion）
FR^N	Freezing for number concentration （freezing）
α_{rs}	$1 - \alpha_{rs}$: Production rate of graupel with the collision of raindrops and snow

There is no exchange term VD_{vc} between water vapor and an cloud water in (**??**), (4.27), and (4.28). The exchange term is calculated with the moist saturation adjustment method. Section 4.2.5 describes about the calculation.

4.2.3 Expression of Particles of Cloud and Precipitation

Particle-size Distribution

When we use a bulk method, we need to give particle-size distribution with a certain suitable function because only mixing ratio, or mixing ratio and number concentration can be directly calculated. A particle-size distribution is needed for the calculations of averaged mass and averaged fall velocity.

The particle-size distribution in the bulk method is often given with the exponential function. This is given by

$$\underbrace{n_x\left(D_x\right)}_{[m^{-4}]} = \underbrace{n_{x0}}_{[m^{-4}]} \exp(\underbrace{-\lambda_x}_{[m^{-1}]} D_x) \tag{4.36}$$

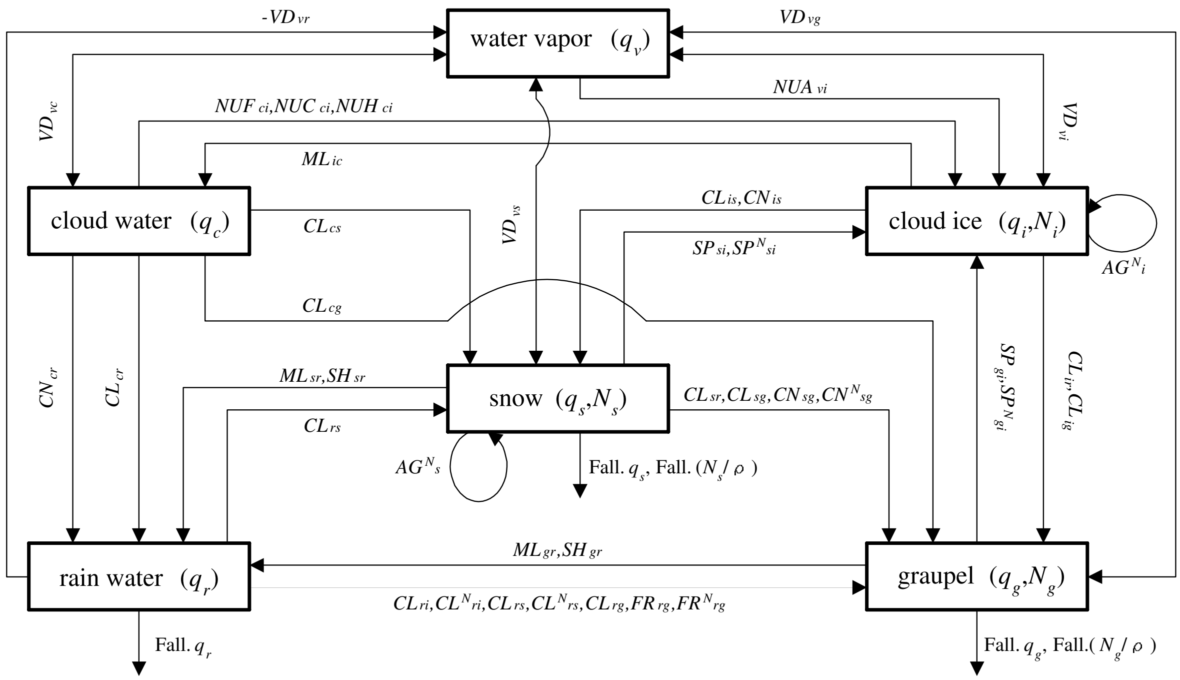

Figure 4.1. Correlation of cloud microphysical processes in a bulk method

using the parameters of an inclination in an exponential function for particle size distribution λ_x and y-section density n_{x0}. The parameters are based on Marshall and Palmer (1948). This distribution is sometimes called the Marshall-Palmer distribution. The distribution is sometimes expressed by using a

gamma function[2]
A gamma function distribution[3] expressing particle size distribution of a category is

$$\underbrace{f_x\left(D_x\right)}_{[m^{-1}]} = \frac{1}{\Gamma(\nu_x)}\left(\frac{D_x}{D_{nx}}\right)^{\nu_x - 1}\frac{1}{D_{nx}}\exp\left(-\frac{D_x}{D_{nx}}\right) \tag{4.43}$$

where D_x is the diameter of particle [m] and $\Gamma\left(\nu_x\right)$ is a standardization (Integral from 0 to ∞ is made to be set to 1.) constant, ν_x is the form parameter of a gamma function and D_{nx} is a characteristic diameter. Number concentrations of particles of cloud and precipitation are expressed as

$$\underbrace{n_x\left(D_x\right)}_{[m^{-4}]} = \underbrace{n_{xt}}_{[m^{-3}]}\underbrace{f_x\left(D_x\right)}_{[m^{-1}]}. \tag{4.44}$$

The averaged diameter of a particle $\bar{D}_x$ in this distribution is

$$\bar{D}_x = \int_0^\infty D_x\ f_x\left(D_x\right)\ dD_x = \frac{\Gamma\left(\nu_x + 1\right)}{\Gamma\left(\nu_x\right)}D_{nx} = \nu_x D_{nx}. \tag{4.45}$$

Here, the relation of (4.38) is used for the last conversion. Generally, the P'th moment of a gamma function distribution is expressed as follows (a complex number is sufficient as P).

[2]A gamma function is one of the special functions which extended Factorial $n!$ even to the complex number. Using this function, we can mechanically solve a certain kind of definite integral. Especially, we often integrated using the exponential function distribution (4.36) in cloud physics. In such a case, definite integral can be solved mechanically. Although there are some definitions for the gamma function $\Gamma\left(x\right)$, it is defined with integration as follows.

$$\Gamma\left(x\right) = \int_0^\infty \exp\left(-t\right)\ t^{x-1}\ dt \tag{4.37}$$

$$\Gamma\left(x+1\right) = x\Gamma\left(x\right) \tag{4.38}$$

$$\Gamma\left(1\right) = 1 \tag{4.39}$$

where especially x is the positive integer NNN, it is set to

$$\Gamma\left(n+1\right) = n\left(n-1\right)\left(n-2\right)\cdots 2\cdot 1\cdot\Gamma\left(1\right) = n! \tag{4.40}$$

and is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{4.41}$$

For example, if this is used instead of carrying out integration by parts 3 times, in the definite integral of cloud physics, the following is mechanically calculable like

$$\int_0^\infty D_x^3\exp\left(-\lambda_x D_x\right)\ dD_x = \frac{1}{\lambda_x^4}\Gamma\left(4\right) = \frac{6}{\lambda_x^4} \tag{4.42}$$

[3]Murakami (1999)

$$\int_0^\infty D_x^P \ f_x\left(D_x\right) \ dD_x = \frac{\Gamma\left(\nu_x + P\right)}{\Gamma\left(\nu_x\right)} D_{nx}^P. \tag{4.46}$$

The exponential function distribution (4.36) is one of the special form of a gamma function distribution (4.43). The distribution (4.43) is the exponential function distribution (4.36) when we set as follows.

$$\nu_x = 1 \tag{4.47}$$

$$D_{nx} = \frac{1}{\lambda_x} \tag{4.48}$$

In this case, (4.46) can be expressed as

$$\int_0^\infty D_x^P \, \lambda_x \exp\left(-\lambda_x D_x\right) \, dD_x = \frac{1}{\lambda_x^P} \Gamma\left(P+1\right). \tag{4.49}$$

It is an availability of a gamma function that definite integral is made simply. The average radius of a particle $\bar{D}_x$ is

$$\bar{D}_x = \frac{1}{\lambda_x}. \tag{4.50}$$

A single distribution is used for an cloud water and cloud ice, and the exponential function distribution[4] is used for precipitation particles of rain water, snow, and graupel. Here, the averaged diameters of cloud water and cloud ice are

$$\bar{D}_c = \left(\frac{6\bar{\rho}q_c}{\pi\rho_w N_c}\right)^{\frac{1}{3}} \tag{4.51}$$

$$\bar{D}_i = \left(\frac{6\bar{\rho}q_i}{\pi\rho_i N_i}\right)^{\frac{1}{3}}. \tag{4.52}$$

where ρ_w is density of cloud water [kg m^{-3}] and ρ_i is density of cloud-ice [kg m^{-3}]. The constant value 1×10^8 m^{-3} is used for the number concentration of cloud water N_c.

Particle size distributions of rain water, snow, and graupel are given as

$$n_r\left(D_r\right) = n_{r0} \exp\left(-\lambda_r D_r\right) \tag{4.53}$$

$$n_s\left(D_s\right) = n_{s0} \exp\left(-\lambda_s D_s\right) \tag{4.54}$$

$$n_g\left(D_g\right) = n_{g0} \exp\left(-\lambda_g D_g\right). \tag{4.55}$$

Note that the unit of n_x is [m^{-4}].

[4]The exponential function distribution is used by Murakami (1999), Ikawa and Saito (1991), Murakami et al. (1994), Ikawa et al. (1991), Lin et al. (1983). Ferrier (1994) uses the exponential function distribution for cloud water and the gamma function distribution for the other precipitation particles.

Averaged Mass and Averaged Fall Velocity

The mass of a particle of cloud or precipitation, which depends on the form of the particle, is given by the empirical equation of an exponentiation of a particle diameter.

$$m_x\left(D_x\right)=\alpha_{ux}D_x^{\beta_{ux}} \tag{4.56}$$

$\beta_{ux}=3$ in the case of a spherical particle. Using a gamma function distribution, the averaged mass is

$$\bar{m}_x=\alpha_{ux}D_{nx}^{\beta_{ux}}\frac{\Gamma\left(\nu_x+\beta_{ux}\right)}{\Gamma(\nu_x)}. \tag{4.57}$$

The fall velocity of a particle is also given by the empirical equation of the exponentiation of a diameter.

$$U_x\left(D_x\right)=\alpha_{ux}D_x^{\beta_{ux}}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\gamma_{ux}} \tag{4.58}$$

In the same way, using a gamma function distribution, the averaged fall velocity of which the weight with the particle number concentration per unit diameter [m^{-4}] is multiplied is

$$\bar{U}_{xN}=\alpha_{ux}D_{nx}^{\beta_{ux}}\frac{\Gamma\left(\nu_x+\beta_{ux}\right)}{\Gamma(\nu_x)}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\gamma_{ux}}. \tag{4.59}$$

The averaged fall velocity of which the weight of mass is multiplied is

$$\bar{U}_{xq}=\alpha_{ux}D_{nx}^{\beta_{ux}}\frac{\Gamma\left(\nu_x+3+\beta_{ux}\right)}{\Gamma\left(\nu_x+3\right)}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\gamma_{ux}}. \tag{4.60}$$

These relations include an exponential function distribution as a special case. Using the relational expression with a gamma function in the cases of (4.47), (4.60) and (4.59) are

$$\bar{U}_{xN}=\alpha_{ux}\frac{\Gamma\left(1+\beta_{ux}\right)}{\lambda_x^{\beta_{ux}}}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\gamma_{ux}} \tag{4.61}$$

$$\bar{U}_{xq}=\alpha_{ux}\frac{\Gamma\left(4+\beta_{ux}\right)}{6\lambda_x^{\beta_{ux}}}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\gamma_{ux}}. \tag{4.62}$$

where ρ_0 is the air density at the surface in the basic state [kg m^{-3}].

In the exponential function distribution[5] (4.53)∼(4.55), the total particle number for categories $x=r,s,g$: N_x is

$$N_x=\int_0^\infty n_{x0}\exp\left(-\lambda_x D_x\right)\,dD_x=\frac{n_{x0}}{\lambda_x}. \tag{4.63}$$

The total mass of water substance x per unit volume is

[5] Ikawa and Saito (1991)

$$\bar{\rho}q_x = \int_0^{\infty} \frac{\pi}{6}\rho_x D_x^3 n_{x0} \exp\left(-\lambda_x D_x\right)\ dD_x = \frac{\pi\rho_x n_{x0}}{\lambda_x^4}. \tag{4.64}$$

From these two equations, an inclination parameter in an exponential function paritlce size distribution λ_x andthe y-section density n_{x0} are

$$\lambda_x = \left(\frac{\pi\rho_x N_x}{\bar{\rho}q_x}\right)^{\frac{1}{3}} \tag{4.65}$$

$$n_{x0} = N_x \left(\frac{\pi\rho_x N_x}{\bar{\rho}q_x}\right)^{\frac{1}{3}}. \tag{4.66}$$

Then, the terminal fall velocity of particle in category x which is weighted by number concentration is

$$\begin{aligned}\bar{U}_{xN} &= \frac{1}{N_x}\int_0^{\infty} U_x\left(D_x\right) n_{x0} \exp\left(-\lambda_x D_x\right)\ dD_x \\ &= \alpha_{ux}\frac{\Gamma\left(1+\beta_{ux}\right)}{\lambda_x^{\beta_{ux}}}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\gamma_{ux}}\end{aligned} \tag{4.67}$$

and this is consistent with (4.61). In the same way, the mass-weighted terminal fall speed of category x is

$$\begin{aligned}\bar{U}_{xq} &= \frac{1}{\bar{\rho}q_x}\int_0^{\infty} \frac{\pi}{6}U_x\left(D_x\right) D_x^3 \rho_x n_{x0} \exp\left(-\lambda_x D_x\right)\ dD_x \\ &= \alpha_{ux}\frac{\Gamma\left(4+\beta_{ux}\right)}{6\lambda_x^{\beta_{ux}}}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\gamma_{ux}}.\end{aligned} \tag{4.68}$$

and this is consistent with (4.62). These averaged fall velocities are used for calculations of change of mixing ratio and number concentration in association with a falling, as described in Section 4.2.6.

Finally, the form parameter of each category is summarized below.

variable	the density of y-section[m^{-4}]	the form parameter of fall speed	density[kg m^{-3}]
q_c	———	$(\alpha_{uc} = 2.98 \times 10^7,\ \beta_{uc} = 2.0,\ \gamma_{uc} = 1.0)$	$\rho_w = 1.0 \times 10^3$
q_r	$n_{r0} = 8.0 \times 10^6$	$\alpha_{ur} = 842,\ \beta_{ur} = 0.8,\ \gamma_{ur} = 0.5$	$\rho_w = 1.0 \times 10^3$
q_i	———	$(\alpha_{ui} = 700,\ \beta_{ui} = 1.0,\ \gamma_{ui} = 0.33)$	$\rho_i = 5.0 \times 10^2$
q_s	$(n_{s0} = 1.8 \times 10^6)$	$\alpha_{us} = 17,\ \beta_{us} = 0.5,\ \gamma_{us} = 0.5$	$\rho_s = 8.4 \times 10^1$
q_g	$(n_{g0} = 1.1 \times 10^6)$	$\alpha_{ug} = 124,\ \beta_{ug} = 0.64,\ \gamma_{ug} = 0.5$	$\rho_g = 3.0 \times 10^2$

4.2.4 Formulizations of Physical Processes of Source terms

Hereafter, we will explain physical processes of source terms. Signs are summarized into table in each section (Sections 4.2.5 and 4.2.7 are the same as this section).

Nuclear Formation of Primary Ice : $NUA_{vi}, NUF_{ci}, NUC_{ci}, NUH_{ci}$

The primary ice nucleation are shown as below.

homogeneous nucleation	sublimation nucleation		water vapor → ice crystals	
	freezing nucleation		supercooled water drops → ice crystals	(
heterogeneous nucleation	sublimation nucleation	sublimation nuclei	water vapor → ice crystals	(
	freezing nucleation	sorption nuclei		
		contact-freezing nuclei	supercooled water drops → ice crystals	(
		immersion-freezing nuclei	supercooled water drops → ice crystals	(

Our model adopts $NUA_{vi}, NUF_{ci}, NUC_{ci}, NUH_{ci}$.

(1)Sublimation Nucleation: NUA_{vi}

(a)Number Concentration of Sublimation Nuclei as a Function of Supercooling Temperature[6]

Number concentration of sublimation nuclei as a function of supercooled temperature T_s is written by

$$NUA_{vi} = \frac{m_{i0}}{\bar{\rho}} \beta_2 N_{i0} \exp\left(\beta_2 T_s\right) \left(\frac{S_i - 1}{S_{wi} - 1}\right)^B \frac{\partial T_s}{\partial z} w \tag{4.69}$$

$$NUA_{vi}^N = \frac{NUA_{vi}}{m_{i0}}. \tag{4.70}$$

Note that a difference in z^* coordinate system (ζ coordinate system) is multiplied by a matrix, as shown in equations (2.58) since the vertical difference is a difference in a real space.

(b)Number Concentration of Sublimation Nuclei as a Function of Supersaturation[7]

Number concentration of sublimation nuclei as a function of supersaturation is written by

$$NUA_{vi} = \frac{m_{i0}}{\bar{\rho}} 15.25 \exp\left(5.17 + 15.25 SS_i\right) \frac{\partial SS_i}{\partial z} w \tag{4.71}$$

$$NUA_{vi}^N = \frac{NUA_{vi}}{m_{i0}} \tag{4.72}$$

Note that a difference in z^* coordinate system (ζ cordinate system) is multiplied by a matrix, as well as the same way in (a).

(c)Considering both Supercooling Temperature and Supersaturation[8]

Ferrier (1994) adopted a method that uses two equations owing to temperature for heterogeneous sublimation nucleation. In the method, the equation of Murakami (1990), Cotton et al. (1986) is used when temperature is more than -5 °C , and that of Meyers et al. (1992) is used when temperature is less than -5 °C .

$$NUA_{vi} = \frac{m_{i0}}{\bar{\rho}} w \frac{\partial N_i}{\partial z} \tag{4.73}$$

$$NUA_{vi}^N = \frac{NUA_{vi}}{m_{i0}} \tag{4.74}$$

[6] Ikawa and Saito (1991), Cotton et al. (1986), Murakami (1990), Ikawa et al. (1991), Murakami et al. (1994), 村上 (1999)
[7] Meyers et al. (1992), 村上 (1999)
[8] Ferrier (1994)

where N_i is divided by -5 °C as follows.

$$N_i = \begin{cases} N_{i01} \exp\left(\beta_2 T_s\right) \left(\dfrac{S_i - 1}{S_{wi} - 1}\right)^B, & T \geq -5\ [°\mathrm{C}] \\ N_{i02} \exp\left(a_1 SS_i - b_1\right), & T < -5\ [°\mathrm{C}] \end{cases} \tag{4.75}$$

Note that a difference in z^* coordinate system (ζ coordinate system) is multiplied by a matrix, as well as the same way in (a) and (b).

Meanings of the signs used in (a)∼(c) are as follows.

a_1	Coefficient of the Ferrier equation in below -5 °C	12.96	
b_1	Coefficient of the Ferrier equation in below -5 °C	0.639	
B	Coefficient of the Huffmann and Vail equation	4.5	
m_{i0}	Minimum mass of cloud ice	10^{-12}	kg
N_{i0}	Coefficient of Fletcher equation	10^{-2}	$\mathrm{m^{-3}}$
N_{i01}	Number of particles of the Ferrier equation in more than -5 °C	10^3	$\mathrm{m^{-3}}$
N_{i02}	Number of particles of the Ferrier equation in below -5 °C	50	$\mathrm{m^{-3}}$
q_{vsi}	Saturation mixing ratio over ice		$\mathrm{kg\ kg^{-1}}$
q_{vsw}	Saturation mixing ratio over water		$\mathrm{kg\ kg^{-1}}$
T	Temperature		K
T_0	Melting point of ice	273.16	K
T_s	Supercooling temperature $(T_0 - T)$		K
S_i	Ratio of mixing ratio of water vapor and saturation mixing ratio of ice in an air parcel		
S_{wi}	Ratio of the saturation mixing ratios of water and ice		
SS_i	Ice supersaturation $(S_i - 1)$		
w	Vertical motion in z coordinate system		$\mathrm{m\ s^{-1}}$
β_2	Coefficient of the Fletcher equation	0.6	$\mathrm{K^{-1}}$
$\bar{\rho}$	Air density in the basic state		$\mathrm{kg\ m^{-3}}$

(2)Immesion Freezing Nucleation: NUF_{ci}

Heterogeneous nucleation of cloud droplets depends on a size, physical and chemical features of freezing nucleus and temperature and size of a cloud drop. We use the nucleation that is extrapolated outside to the size of a cloud droplet with the Bigg (1953) empirical equation[9].

$$NUF_{ci} = B' \left[\exp\left(A' T_s\right) - 1\right] \frac{\bar{\rho} q_c^2}{\rho_w N_c} \tag{4.76}$$

$$NUF_{ci}^N = B' \left[\exp\left(A' T_s\right) - 1\right] \frac{q_c}{\rho_w} \tag{4.77}$$

Meanings of the signs used here are as follows.

A'	Coefficient of the Bigg empirical equation	0.66	$\mathrm{K^{-1}}$
B'	Coefficient of the Bigg empirical equation	100.0	$\mathrm{m^{-3}\ s^{-1}}$
N_c	Number concentration of cloud droplets	1×10^8	$\mathrm{m^{-3}}$
T	Temperature		K
T_0	Melting point of ice	273.16	K

[9]村上 (1999), Ikawa and Saito (1991)

T_s	Supercooled temperature $(T_0 - T)$		K
$\bar{\rho}$	Air density in the basic state		kg m^{-3}
ρ_w	Density of water	1×10^3	kg m^{-3}

(3)Contact Freezing Nucleation： NUC_{ci}

The contact freezing nucleation is occurred with a collision of an ice particle core (freezing core) and a supercooled cloud droplet. The following three processes are considered to it.

- Brownian diffusion
- diffusiophoresis
- thermophoresis

The generation rate of ice particles by these processes is given by

$$\left[\frac{dN_c}{dt}\right]_b = F_1 \mathcal{D}_{ar} \tag{4.78}$$

$$\left[\frac{dN_c}{dt}\right]_v = F_1 F_2 \frac{R_v T}{\mathcal{L}_v} \tag{4.79}$$

$$\left[\frac{dN_c}{dt}\right]_t = F_1 F_2 f_t \tag{4.80}$$

where

$$F_1 = 2\pi D_c N_c N_{ar} \tag{4.81}$$

$$F_2 = \frac{\kappa_a}{p}\left(T - T_{cl}\right) \tag{4.82}$$

$$f_t = \frac{0.4\left[1 + 1.45K_n + 0.4\exp\left(-1/K_n\right)\right]\left(\kappa + 2.5K_n\kappa_a\right)}{\left(1 + 3K_n\right)\left(2\kappa + 5\kappa_a K_n + \kappa_a\right)}. \tag{4.83}$$

K_n in the equations is the Knudsen number which is defined as

$$K_n = \lambda_{a0}\frac{p_{00}}{T_{00}R_a}\frac{T}{p}. \tag{4.84}$$

The diffusion coefficient of an aerosol particle is

$$\mathcal{D}_{ar} = \frac{kT_{cl}}{6\pi R_a \mu}\left(1 + K_n\right). \tag{4.85}$$

The number concentration N_{ar} of the contact freezing nuclei activated by temperature T_{cl}[10] is

$$N_{ar} = N_{a0}\left(270.16 - T_{cl}\right)^{1.3}. \tag{4.86}$$

The formation of an ice particle with the contact freezing nuclei is finally represented by the sum of the three processes, as shown in follows.

$$NUC_{ci}^{N} = \frac{1}{\bar{\rho}}\left(\left[\frac{dN_c}{dt}\right]_b + \left[\frac{dN_c}{dt}\right]_v + \left[\frac{dN_c}{dt}\right]_t\right) \tag{4.87}$$

$$NUC_{ci} = \frac{\bar{\rho}q_c}{N_c}NUC_{ci}^{N} \tag{4.88}$$

Meanings of the signs used here are as follows.

D_c	Diameter of a cloud droplet		m
k	The Boltzmann number	1.380658×10^{-23}	J K^{-1}
$\mathcal{L}_v$	Latent heat of evaporation of water		J kg^{-1}
N_{a0}	Coefficient in the number concentration of contact freezing nuclei	2×10^{5}	m^{-3}
N_c	Number concentration of cloud droplets	1×10^{8}	m^{-3}
p	Pressure		Pa
p_{00}	Standard pressure	101325	Pa
R_a	Radius of an aerosol particle	3×10^{-7}	m
R_v	The gas constant of water vapor	461.0	J K^{-1} kg^{-1}

[10] Cotton et al. (1986)

T	Temperature		K
T_{00}	Temperature in the basic state	293.15	K
T_{cl}	Temperature of a cloud droplet		K
κ	Thermal conductivity of air	2.4×10^{-2}	J m^{-1} s^{-1} K^{-1}
κ_a	Thermal conductivity of an aerosol particle		J m^{-1} s^{-1} K^{-1}
λ_{a0}	Mean free path at p_{00}, T_{00}	$6.6{\times}10^{-8}$	m
μ	Coefficient of viscosity of air		kg m^{-1} s^{-1}
$\bar{\rho}$	Air density in the basic state		kg m^{-3}

(4)Homogeneous Freezing Nuclei： NUH_{ci}

We assume that cloud water freezes momentarily when temperature is below -40 °C [11]. Rate of the homogeneous freezing nuclei is written by

$$NUH_{ci}^{N} = \frac{1}{\bar{\rho}} \frac{N_c}{2\Delta t} \tag{4.89}$$

$$NUH_{ci} = \frac{q_c}{2\Delta t} \tag{4.90}$$

where Δt is the time step in the leap frog method. Note that generation of cloud water must be given by the moist saturation adjustment before this calculation (Ferrier, 1994).

Secondary Nucleation of Ice Crystals: SP

Although there are still many points which are not known about the nucleation of secondary ice crystals, the following processes are mainly known.

- While snow and graupel grow up with collection of cloud droplets, minute particles of ice are generated (Hallett and Mossop, 1974).
- Snow and graupel collide each other while they are falling, and the fragment of small ice is generated (Vardiman, 1978).
- When the large supercooled water droplet, which is generated near the cloud top, freezes, ice crystals of high-concentration are generated (Hobbs and Rangno, 1985).

We consider only the Hallett-Mossop rime splintering mechanism[12]. Generation rates of the secondary nucleation of ice crystals are

$$SP_{si}^{N} = \frac{1}{\bar{\rho}} \times 3.5 \times 10^8 f\left(T_s\right) CL_{cs} \tag{4.91}$$

$$SP_{si} = m_{i0} SP_{si}^{N} \tag{4.92}$$

$$SP_{gi}^{N} = \frac{1}{\bar{\rho}} \times 3.5 \times 10^8 f\left(T_g\right) CL_{cg} \tag{4.93}$$

$$SP_{gi} = m_{i0} SP_{gi}^{N}. \tag{4.94}$$

Note that the secondary nucleation of ice crystals has never occurred in the wet growth of graupel. Function of the temperature of a particle $f\left(T_x\right)$ (T_x is T_s or T_g) is defined as follows.

[11] Ikawa and Saito (1991), Ferrier (1994)
[12] 村上 (1999), Ikawa and Saito (1991), Cotton et al. (1986)

$$f(T_x) = \begin{cases} 0, & T_x > 270.16\ [K] \\ \dfrac{T_x - 268.16}{2}, & 268.16 \leq T_x \leq 270.16\ [K] \\ \dfrac{268.16 - T_x}{3}, & 265.16 \leq T_x \leq 268.16\ [K] \\ 0, & T_x < 265.16\ [K] \end{cases} \tag{4.95}$$

Although $f(T_x)$ will become a negative value in (72) of Cotton et al. (1986), if an absolute value is taken, it will become the same as the equation (4.95). Although (4.95) is a function that it may be $f(T_x) = 0$ at $T_x = 268.16$ [K], Ikawa et al. (1991), Ikawa and Saito (1991) use a function that it may be $f(T_x) = 1$ at $T_x = 268.16$[K] as follows.

$$f(T_x) = \begin{cases} 0, & T_x \geq 270.16\ [K] \\ \dfrac{270.16 - T_x}{2}, & 268.16 < T_x < 270.16\ [K] \\ 1, & T_x = 268.16\ [K] \\ \dfrac{T_x - 265.16}{3}, & 265.16 \leq T_x < 268.16\ [K] \\ 0, & T_x < 265.16\ [K] \end{cases} \tag{4.96}$$

This function has the maximum value at -5 °C and is considered to be more reasonable.

Meanings of the signs used here are as follows.

CL_{cg}	Growth rate of graupel in the collision with cloud droplets		s^{-1}
CL_{cs}	Growth rate of snow in the collision with cloud droplets		s^{-1}
m_{i0}	The mass of the minimum cloud ice	10^{-12}	kg
T_s	Temperature of snow		K
T_g	Temperature of graupel		K
T_x	Temperature of snow or graupel		K
$\bar{\rho}$	Air density in the basic state		kg m^{-3}

Water Vapor Diffusion Growth : VD

Diffusion growth is the generation, the growth, disappearance, and consumption by exchange of the water molecule between moisture and a particle. There are the following processes and some of them are considered in our model.

gaseous phase − liquid phase	condensation	water vapor → cloud water	calculated by the moist saturation a
		water vapor → rain water	neglected because of small amount
	evaporation	cloud water → water vapor	calculated by the moist saturation a
		rain water → water vapor	$VD_{vr} < 0$
gaseous phase − solid phase	sublimation	water vapor → cloud ice	$VD_{vi} > 0$
		water vapor → snow	$VD_{vs} > 0$
		water vapor → graupel	$VD_{vg} > 0$
		water vapor → hail	$VD_{vh} > 0$
	sublimation evaporation	cloud ice → water vapor	$VD_{vi} < 0$
		snow → water vapor	$VD_{vs} < 0$

		graupel → water vapor	$VD_{vg} < 0$
		hail → water vapor	$VD_{vh} < 0$

(1)Evaporation of Rain Water： VD_{rv}

Variations of mixing ratio and number concentration of rain water [13] are represented by

$$VD_{vr} = -VD_{rv} = \frac{2\pi}{\bar{\rho}}\left(S_w - 1\right) G_w\left(T, p\right) VENT_r \tag{4.97}$$

where

$$G_w\left(T, p\right) = \left(\frac{\mathcal{L}_v^2}{\kappa R_v T^2} + \frac{1}{\bar{\rho} q_{vsw} \mathcal{D}_v}\right)^{-1}. \tag{4.98}$$

The term of the ventilation coefficient of rain water is represented by

$$VENT_r = n_{r0}\left[0.78\lambda_r^{-2} + 0.31 S_c^{\frac{1}{3}} \nu^{-\frac{1}{2}} \alpha_{ur}^{\frac{1}{2}} \Gamma\left(\frac{5+\beta_{ur}}{2}\right) \lambda_r^{-\frac{(5+\beta_{ur})}{2}} \left(\frac{\rho_0}{\bar{\rho}}\right)^{\frac{1}{4}}\right]. \tag{4.99}$$

(2)Sublimation Condensation of Snow and Graupel (Sublimation Evaporation)： VD_{vs}, VD_{vg} [14]

Sublimation condensation (sublimation evaporation) rates of snow and graupel are formulized like rain water. However, they are divided in the cases more and less than the melting point of ice. Here, snow and graupel are simultaneously formulized as $x = s, g$. That is,

considering the latent heat of freezing in a collecting supercooled cloud droplets at $T < T_0$,

$$VD_{vx} = \frac{2\pi}{\bar{\rho}}\left(S_i - 1\right) G_i\left(T, p\right) VENT_x - \frac{\mathcal{L}_s \mathcal{L}_f}{\kappa R_v T^2} G_i\left(T, p\right) CL_{cx}. \tag{4.100}$$

At $T > T_0$,
when a melting process occurs,

$$VD_{vx} = 2\pi \mathcal{D}_v\left(q_v - q_{vs}\left(T_0\right)\right) VENT_x \tag{4.101}$$

when a melting process does not occur,

$$VD_{vx} = \frac{2\pi}{\bar{\rho}}\left(S_w - 1\right) G_w\left(T, p\right) VENT_x. \tag{4.102}$$

[13] 村上 (1999)
[14] 村上 (1999), Ikawa and Saito (1991), Lin et al (1983)

where

$$G_i\left(T,p\right)=\left(\frac{\mathcal{L}_s^2}{\kappa R_v T^2}+\frac{1}{\bar{\rho}q_{vsi}\mathcal{D}_v}\right)^{-1} \tag{4.103}$$

where $G_w\left(T,p\right)$ is given by the equation (4.98). A ventilation coefficient at $x=s,g$ is as follows.

$$VENT_x=n_{x0}\left[0.78\lambda_x^{-2}+0.31S_c^{\frac{1}{3}}\nu^{-\frac{1}{2}}\alpha_{ux}^{\frac{1}{2}}\Gamma\left(\frac{5+\beta_{ux}}{2}\right)\lambda_x^{-\frac{(5+\beta_{ux})}{2}}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\frac{1}{4}}\right] \tag{4.104}$$

(3)Growth of Cloud Ice in Gaseous Phase： VD_{vi} [15]

Variation of growth of cloud ice in gaseous phase is represented as follows.

$$VD_{vi}=\frac{q_v-q_{vsi}}{q_{vsw}-q_{vsi}}a_1\left(\bar{m}_i\right)^{a_2}\frac{N_i}{\bar{\rho}} \tag{4.105}$$

where a_1,a_2 is a parameter depending on temperature (Köenig (1971)) and $\bar{m}_i$ is mean mass of cloud ice given as following equation.

$$\bar{m}_i=\frac{q_i\bar{\rho}}{N_i} \tag{4.106}$$

[15] Ikawa and Saito (1991), Ikawa et al. (1991)

Meanings of the signs used in (1)-(3) are as follows.

CL_{cg}	Growth rate of graupel in coalescence with cloud droplets		s^{-1}
CL_{cs}	Growth rate of snow in coalescence with cloud droplets		s^{-1}
$\mathcal{D}_v$	Diffusion coefficient of water vapor		$m^2\ s^{-1}$
$\mathcal{L}_f$	Latent heat of melting of water		$J\ kg^{-1}$
$\mathcal{L}_s$	Latent heat of sublimation of water		$J\ kg^{-1}$
$\mathcal{L}_v$	Latent heat of evaporation of water		$J\ kg^{-1}$
n_{g0}	Density of graupel in the y section		m^{-4}
n_{r0}	Density of rain water in the y section	8.0×10^6	m^{-4}
n_{s0}	Density of snow in the y section		m^{-4}
$q_{vs}(T_0)$	Saturation mixing ratio of the melting point of water		$kg\ kg^{-1}$
q_{vsi}	Ice saturation mixing ratio		$kg\ kg^{-1}$
q_{vsw}	Water saturation mixing ratio		$kg\ kg^{-1}$
R_v	The gas constant of water vapor	461.0	$J\ K^{-1}\ kg^{-1}$
S_c	Schmidt number	0.6	
$S_i - 1$	Ice supersaturation of an air parcel		
$S_w - 1$	Water supersaturation of an air parcel		
T	Temperature		K
T_0	Melting point of ice	273.16	K
α_{ug}	Coefficient in the relational expression of fall velocity and diameter of graupel	124	$m^{1-\beta_{ug}}\ s^{-1}$
α_{ur}	Coefficient in the relational expression of fall velocity and diameter of rain water	842	$m^{1-\beta_{ur}}\ s^{-1}$
α_{us}	Coefficient in the relational expression of fall velocity and diameter of snow	17	$m^{1-\beta_{us}}\ s^{-1}$
β_{ug}	Coefficient in the relational expression of fall velocity and diameter of graupel	0.64	
β_{ur}	Coefficient in the relational expression of fall velocity and diameter of rain water	0.8	
β_{us}	Coefficient in the relational expression of fall velocity and diameter of snow	0.5	
λ_g	Inclination in a reversed exponential function expressing particle size distribution of graupel		m^{-1}
λ_r	Inclination in a reversed exponential function expressing particle size distribution of rain water		m^{-1}
λ_s	Inclination in a reversed exponential function expressing particle size distribution of snow		m^{-1}
κ	Thermal conductivity of air	2.4×10^{-2}	$J\ m^{-1}\ s^{-1}\ K^{-1}$
ν	Coefficient of kinematic viscosity of air		$m^2\ s^{-1}$
$\bar{\rho}$	Air density in the basic state		$kg\ m^{-3}$
ρ_0	Density at the surface in the basic state		$kg\ m^{-3}$

Particle Collection: CL

In this section, we formulize the growth process of the particle which captures a certain particle by the collision between various particles. For the particle collision, the following processes can be respectively considered to the categories of the particles which are defined before.

process	sink	source	sign	meaning	model
Collection of cloud water	q_c	q_i	CL_{ci}	Growth of cloud ice by collection with cloud water	×
	q_c	q_s	CL_{cs}	Growth of snow by collection with cloud water	○
	q_c	q_g	CL_{cg}	Growth of graupel by collection with cloud water	○
	q_c	q_h	CL_{ch}	Growth of hail by collection with cloud water	△
	q_c	q_r	CL_{cr}	Growth of rain water by collection with cloud water	○

Collection of rain water	q_r	q_i	CL_{ri}	Growth of cloud ice by collection with rain water	◯
	q_r	q_s	CL_{rs}	Growth of snow by collection with rain water	◯
	q_r	q_g	CL_{rg}	Growth of graupel by collection with rain water	◯
	q_r	q_h	CL_{rh}	Growth of hail by collection with rain water	△
Collection of cloud ice	q_i	q_r	CL_{ir}	Growth of rain water by collection with cloud ice	◯
	q_i	q_s	CL_{is}	Growth of snow by collection with cloud ice	◯
	q_i	q_g	CL_{ig}	Growth of graupel by collection with cloud ice	◯
	q_i	q_h	CL_{ih}	Growth of hail by collection with cloud ice	△
Collection of snow	q_s	q_r	CL_{sr}	Growth of rain water by collection with snow	◯
	q_s	q_g	CL_{sg}	Growth of graupel by collection with snow	◯
	q_s	q_h	CL_{sh}	Growth of hail by collection with snow	△

In this table, "◯" and "×" denote the signs of which it is included or is not included in our model, respectively, and "△" is the sign of suspending for including in our model at present.

The number concentration will change for the collection of snow particles (aggregation) and that of rain water (accretion) but the mixing ratio do not change. These are described in the other section.

In the following section, we formulize the coalescences between large particles (rain water, snow, and graupel), which have a sufficient value of fall velocity, a large particle and a small particle (cloud water and cloud ice) of which its fall velocity can be neglected, and the growth of graupel by collection of supercooled rain water and ice crystals[16].

(1)Coalescence between Rain Water, Snow and Graupel： CL_{xy} $(x, y = r, s, g; x \neq y)$

Variations of mixing ratio and number concentration by the collection between precipitation particles (rain water, snow, and hail) are formulized as follows.

$$CL_{xy} = \pi^2 \frac{\rho_x}{\bar{\rho}} E_{xy} \sqrt{\left(\bar{U}_x - \bar{U}_y\right)^2 + \alpha \bar{U}_x \bar{U}_y} n_{x0} n_{y0} \left(\frac{5}{\lambda_x^6 \lambda_y} + \frac{2}{\lambda_x^5 \lambda_y^2} + \frac{0.5}{\lambda_x^4 \lambda_y^3} \right) \quad (4.107)$$

$$CL_{xy}^N = \frac{\pi}{2\bar{\rho}} E_{xy} \sqrt{\left(\bar{U}_x - \bar{U}_y\right)^2 + \alpha \bar{U}_x \bar{U}_y} n_{x0} n_{y0} \left(\frac{1}{\lambda_x^3 \lambda_y} + \frac{1}{\lambda_x^2 \lambda_y^2} + \frac{1}{\lambda_x \lambda_y^3} \right) \quad (4.108)$$

where $x, y = r, s, g; x \neq y$. Meanings of the signs used are as follows.

E_{xy}	Collection efficiency of a particle		
n_{x0}	Density of a category x in the y section		m^{-4}
n_{y0}	Density of a category y in the y section		m^{-4}
$\bar{U}_x$	Mean fall velocity weighted by a mass of a category x		$\mathrm{m\ s}^{-1}$
$\bar{U}_y$	Mean fall velocity weighted by mass of a category y		$\mathrm{m\ s}^{-1}$
α	Coefficient of an adjustment term	0.04	
λ_x	Inclination in a reversed exponential function expressing particle size distribution of a category x		m^{-1}
λ_y	Inclination in a reversed exponential function expressing particle size distribution of a category y		m^{-1}
$\bar{\rho}$	Air density in the basic state		$\mathrm{kg\ m}^{-3}$
ρ_x	Density of a particle of a category x		$\mathrm{kg\ m}^{-3}$

[16]Murakami (1999), Lin et al. (1983), Murakami (1990), Ikawa and Saito (1991)

(2)Collection of Rain water, Snow, and Graupel with Cloud Water and Cloud Ice: CL_{cy}, CL_{iy} $(y = r, s, g)$

Because fall velocities of cloud water and cloud ice are relatively small to those of rain water, snow and graupel, the collection is represented as follows.

$$CL_{xy} = \frac{\pi}{4}\bar{E}_{xy}\, n_{y0}\, q_x\, \alpha_{uy}\, \Gamma\left(3+\beta_{uy}\right)\, \lambda_y^{-(3+\beta_{uy})}\, \left(\frac{\rho_0}{\bar{\rho}}\right)^{\frac{1}{2}} \tag{4.109}$$

The mean collection efficiency for coalescence with cloud ice $\bar{E}_{iy}$ is constant, and that of coalescence with cloud water $\bar{E}_{cy}$ is given by

$$\bar{E}_{cy} = \frac{Stk^2}{\left(Stk+0.5\right)^2} \tag{4.110}$$

where Stk is the Stokes number which is given with the average radius of cloud water, cloud ice, and precipitation particle. Ikawa and Saito (1991) calculates as follows.

$$Stk = \bar{D}_c^2 \rho_w \frac{\bar{U}_y}{9\mu \bar{D}_y} \tag{4.111}$$

Meanings of the signs used here are as follows.

D_c	Diameter of a particle of cloud water		m
D_y	Diameter of a particle of a category y		m
n_{y0}	Density of a category y in the y section		m^{-4}
$\bar{U}_y$	Mean fall velocity multiplied by an weight coefficient of mass of a category y		$\mathrm{m\ s}^{-1}$
α_{uy}	Coefficient in the relational expression of fall velocity and diameter of a category y		$\mathrm{m}^{1-\beta_{uy}}\ s^{-1}$
β_{uy}	Coefficient in the relational expression of fall velocity and diameter of a category y		
λ_y	Inclination in a reversed exponential function expressing particle size distribution of a category y		m^{-1}
μ	Coefficient of viscosity of air		$\mathrm{kg\ m^{-1}\ s^{-1}}$
$\bar{\rho}$	Air density in the basic state		$\mathrm{kg\ m^{-3}}$
ρ_0	Density at the surface in the basic state		$\mathrm{kg\ m^{-3}}$
ρ_w	Density of water	1×10^3	$\mathrm{kg\ m^{-3}}$

(3)Formation Process of Graupel by Collection of a Supercooled Raindrop and Ice Crystals: CL_{ri}

When it is assumed that a raindrop freezes instantaneously, variations of mixing ratio and number concentration of graupel with its formation by the collision of a supercooled raindrop and ice crystals are given as follows[17].

$$CL_{ri} = \frac{\pi^2}{24}E_{ir}\, N_i\, n_{r0}\, \alpha_{ur}\, \Gamma\left(6+\beta_{ur}\right)\, \lambda_r^{-(6+\beta_{ur})}\, \left(\frac{\rho_0}{\bar{\rho}}\right)^{\frac{1}{2}} \tag{4.112}$$

[17] 村上 (1999)

$$CL_{ri}^{N} = \frac{\pi}{4\bar{\rho}} E_{ir}\ N_i\ n_{r0}\ \alpha_{ur}\ \Gamma\left(3+\beta_{ur}\right)\ \lambda_r^{-(3+\beta_{ur})} \left(\frac{\rho_0}{\bar{\rho}}\right)^{\frac{1}{2}} \tag{4.113}$$

Meanings of the signs used here are as follows.

E_{ir}	Collection efficiency of a particle	1.0	
n_{r0}	Density of rain water in the y section	8.0×10^6	m^{-4}
α_{ur}	Coefficient in the relational expression of fall velocity and diameter of rain water	842	$\mathrm{m}^{1-\beta_{ur}}\ \mathrm{s}^{-1}$
β_{ur}	Coefficient in the relational expression of fall velocity and diameter of rain water	0.8	
λ_r	Inclination in a reversed exponential function expressing particle size distribution of rain water		m^{-1}
$\bar{\rho}$	Air density in the basic state		$\mathrm{kg\ m}^{-3}$
ρ_0	Density at the surface in the basic state		$\mathrm{kg\ m}^{-3}$

(4)Summary of Collection Efficiencies for Coalescences： E_{xy} [18]

Collection efficiencies E_{xy} for coalescences between particles shown in (1)-(3) are summarized in the following table.

E_{cr}	Collection efficiency for the coalescence of rain water with cloud water	$Stk^2 \Big/ (Stk+0.5)^2$
E_{cs}	Collection efficiency for the coalescence of snow with cloud water	$Stk^2 \Big/ (Stk+0.5)^2$
E_{cg}	Collection efficiency for the coalescence of graupel with cloud water	$Stk^2 \Big/ (Stk+0.5)^2$
E_{rs}	Collection efficiency for the coalescence of snow with rain water	1.0
E_{rg}	Collection efficiency for the coalescence of graupel with rain water	1.0
E_{ir}	Collection efficiency for the coalescence of rain water with cloud ice	1.0
E_{is}	Collection efficiency for the coalescence of snow with cloud ice	1.0
E_{ig}	Collection efficiency for the coalescence of graupel with cloud ice	0.1
E_{sr}	Collection efficiency for the coalescence of rain water with snow	1.0
E_{sg}	Collection efficiency for the coalescence of graupel with snow	0.001

(5)Category Allocation Ratios of Snow and Raindrop after the Coalescence： α_{rs}

In the coalescence of snow and raindrop, it is difficult that we decide an adequate category a particle formed by the coalescence in the layer less than 0 °C . Here, the allocation ratios of the snow and graupel, which appear in the equations (4.31) and (4.32), is given by using the average masses of rain water $\bar{m}_r$ and snow $\bar{m}_s$ as follows.

$$\alpha_{rs} = \frac{\bar{m}_s^2}{\bar{m}_s^2 + \bar{m}_r^2} \tag{4.114}$$

[18]Ikawa and Saito (1991), Ikawa et al. (1991)

The average masses of rain water $\bar{m}_r$ and snow $\bar{m}_s$ are given as follows.

$$\bar{m}_r = \rho_r \left(\frac{4}{\lambda_r}\right)^3 \tag{4.115}$$

$$\bar{m}_s = \rho_s \left(\frac{4}{\lambda_s}\right)^3 \tag{4.116}$$

Using these equations, we can find that the generation ratio of graupel by the coalescence of a raindrop and snow is $(1-\alpha_{rs})$. Note that the ratios in (4.114) are tentative values.

Production of Graupel： PG

Production of graupel is an important and complicated problem, since graupel is producted by the collision between other particles. In this section, production process of graupel is explained in detail especially, although it should be explained in the section of "Collision between particles".

Production of graupel has two processes: dry growth and wet growth. By the former process, all supercooled cloud droplets freeze in an instant on contact with graupel, and surface of graupel remains dry. By the latter process, not all supercooled cloud droplets freeze by the latent heat which supercooled cloud droplets emit when they collided with graupel. Surface of graupel gets wet.

In dry growth process, although all supercooled cloud droplets contribute to growth, cloud ice and snow hardly contribute to growth because the collection efficiency of cloud ice or snow colliding with graupel is small. On the contrary, in wet growth process, a freezing amount of supercooled water is decided by the budget of sensible and latent heat. Cloud ice or snow are efficiently captured by graupel.

Dry growth process is defined as following[19].

$$PG_{dry} = CL_{cg} + CL_{rg} + CL_{ig} + CL_{sg} \tag{4.117}$$

On the contrary, wet growth process is defined as following.

$$PG_{wet} = \frac{2\pi\left[\kappa T_s + \mathcal{L}_v \mathcal{D}_v \bar{\rho}\left(q_{vs}\left(T_0\right) - q_v\right)\right]}{\bar{\rho}\left(\mathcal{L}_f - C_w T_s\right)} VENT_g + \left(CL'_{ig} + CL'_{sg}\right)\left(1 + \frac{C_i T_s}{\mathcal{L}_f - C_w T_s}\right) \tag{4.118}$$

Where $VENT_g$ is ventilation efficiency (4.104). Whether graupel grows by dry growth process or wet growth process is decided by following relation of PG_{dry} and PG_{wet}.

$$PG_g = PG_{dry}, \qquad PG_{dry} \le PG_{wet} \tag{4.119}$$

$$PG_g = PG_{wet}, \qquad PG_{dry} > PG_{wet} \tag{4.120}$$

Meanings of signs used in this section are shown as follows.

CL_{cg}	Growth rate of cloud water by coalescence with cloud water	s^{-1}
CL_{ig}	Growth rate of cloud ice by coalescence with cloud water	s^{-1}
CL'_{ig}	Growth rate of cloud ice by coalescence with cloud water in wet growth process	s^{-1}
CL_{rg}	Growth rate of rain by collecting cloud water	s^{-1}
CL_{sg}	Growth rate of snow by collecting cloud water	s^{-1}

[19] 村上 (1999)

CL'_{sg}	Growth rate of snow by collecting cloud water in wet growth process		s^{-1}
C_i	Specific heat at constant pressure of ice	2.0×10^3	$J\ K^{-1}kg^{-1}$
C_w	Specific heat at constant pressure of water	4.17×10^3	$J\ K^{-1}kg^{-1}$
$\mathcal{D}_v$	Diffusion coefficient of water vapor		$m^2\ s^{-1}$
$\mathcal{L}_f$	Latent heat of fusion of water		$J\ kg^{-1}$
$\mathcal{L}_v$	Latent heat of evaporation of water		$J\ kg^{-1}$
$q_{vs}(T_0)$	Saturation mixing ratio at melting point of water point of water		$kg\ kg^{-1}$
T	Temperature		K
T_0	Melting point of ice	273.16	K
T_s	Temperature in supercooling $(T_0 - T)$		K
κ	Efficiency of conduction of heat of air	2.4×10^{-2}	$J\ m^{-1}\ s^{-1}\ K^{-1}$
$\bar{\rho}$	Air Density in the basic state		$kg\ m^{-3}$

Aggregation： AG

Aggregation, as well as production of graupel, should be categorized in "Collision between particles," however, we explain it especially in this section.

We consider the following two phenomenons for aggregation. First, the number of cloud ice per unit volume decreases by the collision between cloud ices (ice crystals). Second, the number of snow per unit volume decreases by the collision between snows (snowflakes).

By aggregation process, only the number concentration changes but mixing ratio does not change.

(1)Aggregation between Cloud Ice Particles： AG_i^N [20]

Decrease of the number concentration of cloud ices (ice crystals) by aggregation process is defined as following.

$$AG_i^N = \left[\frac{d}{dt}\left(\frac{N_i}{\bar{\rho}}\right)\right]_{aggr} = -\frac{c_1}{2\bar{\rho}}N_i \tag{4.121}$$

where c_1 is defined as following.

$$c_1 = \frac{\bar{\rho}q_i\alpha_{ui}E_{ii}X}{\rho_i}\left(\frac{\rho_0}{\bar{\rho}}\right)^{\frac{1}{3}} \tag{4.122}$$

Meanings of signs used in this section are shown as follows.

E_{ii}	Collection efficiency between ice crystals	0.1	
X	Spectrum dispersion of fall velocity of ice crystal	0.25	
α_{ui}	The coefficient which appears in the equation of fall velocity and diameter of ice crystal	700	$m^{1-\beta_{ui}}\ s^{-1}$
β_{ui}	The coefficient which appears in the equation of fall velocity and diameter of ice crystal	1.0	
$\bar{\rho}$	Air density in the basic state		$kg\ m^{-3}$
ρ_0	Air density at the surface in the basic state		$kg\ m^{-3}$
ρ_i	Density of cloud ice	5.0×10^2	$kg\ m^{-3}$

[20]村上 (1999), Ikawa and Saito (1991)

(2)Aggregation between Snow Particles： AG_s^N [21]

A decrease of the number concentration of snow (snowflakes) by aggregation process is defined as following. By this process, only the number concentration of snow (N_s) changes but mixing ratio of snow (q_s) does not change.

$$AG_s^N = \left[\frac{d}{dt}\left(\frac{N_s}{\bar{\rho}}\right)\right]_{aggr} = -\frac{1}{\bar{\rho}}\frac{\alpha_{us}E_{ss}I\left(\beta_{us}\right)}{4\times 720}\pi^{\frac{1-\beta_{us}}{3}}\bar{\rho}^{\frac{2+\beta_{us}}{3}}\rho_s^{\frac{-2-\beta_{us}}{3}}q_s^{\frac{2+\beta_{us}}{3}}N_s^{\frac{4-\beta_{us}}{3}} \tag{4.123}$$

where $I\left(\beta_{us}\right)$ is defined as following.

$$I\left(\beta_{us}\right) = \int_0^\infty\int_0^\infty x^3y^3\left(x+y\right)^2\left|x_{us}^\beta - y_{us}^\beta\right|\exp\left[-\left(x+y\right)\right]\,dxdy \tag{4.124}$$

This equation is solved with Gauss's hypergeometric function as following. [22]

$$I\left(\beta_{us}\right) = \Gamma\left(\beta_{us}\right)2^{1-d}\sum_{i=1}^{3}C_i\left[\frac{F\left(1,d;8-i;0.5\right)}{7-i} - \frac{F\left(1,d;4+\beta_{us};0.5\right)}{3+\beta_{us}+i}\right] \tag{4.129}$$

where

$$\begin{aligned} d &= 10+\beta_{us} \\ C_1 &= 1 \\ C_2 &= 3 \\ C_3 &= 1. \end{aligned}$$

We adopted the following values as typical values of $I\left(\beta_{us}\right)$ (Ikawa and Saito, 1991; Mizuno, 1990).

β_{us}	0.4	0.5	0.6
$I\left(\beta_{us}\right)$	1108	1610	2566

[21] 村上 (1999), Ikawa and Saito (1991)

[22] The hypergeometric function is one of solutions of hypergeometric equation which is two order linear ordinary differential equation, and has regular sigularity at $x = 0, 1, \infty$.

$$F\left(x,a;b;c\right) = 1 + \frac{a\cdot b}{c}\frac{x}{1!} + \frac{a\left(a+1\right)b\left(b+1\right)}{c\left(c+1\right)}\frac{x^2}{2!} + \cdots \tag{4.125}$$

where $c \neq 0, -1, -2, \cdots$. With Pockhimer's sign,

$$\left(a\right)_n = a\left(a+1\right)\left(a+2\right)\cdots\left(a+n-1\right) = \frac{\left(a+n-1\right)!}{\left(a-1\right)!} \tag{4.126}$$

$$\left(a\right)_0 = 1 \tag{4.127}$$

$$F\left(x,a;b;c\right) = \sum_{n=0}^{\infty}\frac{\left(a\right)_n\left(b\right)_n}{\left(c\right)_n}\frac{x^n}{n!} \tag{4.128}$$

As special limit of this hypergeometric function, there is confluent hypergeometric function

Meanings of signs used in these equations are shown as follows.

E_{ss}	Collection efficiency between snows	0.1	
α_{us}	The coefficient which appears in the equation of fall velocity and diameter of snow	17	$\mathrm{m}^{1-\beta_{us}}\ \mathrm{s}^{-1}$
β_{us}	The coefficient which appears in the equation of fall velocity and diameter of snow	0.5	
$\bar{\rho}$	Air density in the basic state		kg m^{-3}
ρ_s	Density of snow	8.4×10^1	kg m^{-3}

Conversion： CN

Particle conversion from a certain category 'x' into other category 'y' is expressed by CN_{xy}.

For example, Kessler (1969)'s expression of conversion from cloud water into rain water is one of the earliest formulation of category conversion.

We consider the following types of category conversions.

process	consumed particle	growing particle	sign	meaning
conversion from cloud water to rain water	q_c	q_r	CN_{cr}	by aggregation
conversion from cloud ice to snow	q_i	q_s	CN_{is}	by aggregation and sublimation
conversion from snow to graupel	q_s	q_g	CN_{sg}	by growth with capturing cloud
conversion from snow to hail	q_s	q_h	CN_{sh}	by growth with capturing cloud
conversion from graupel to hail	q_g	q_h	CN_{gh}	by growth with capturing cloud
conversion from snow to graupel	q_s	q_g	CN_{sg}	by freezing of rain water
conversion from snow to hail	q_s	q_h	CN_{sh}	by freezing of rain water
conversion from graupel to hail	q_g	q_h	CN_{gh}	by freezing of rain water

Hereafter, we consider about conversion processes of $CN_{cr}, CN_{is}, CN_{sg}$.

(1)Conversion from Cloud to Rain： CN_{cr} [23]

The conversion from cloud water to rain water is based on coalescence growth of cloud droplets. Kessler (1969) parameterized this process first. This process is investigated in detail by Berry (1968),and Berry and Reinhardt (1974). Recently, the method for the conversion based on this process is used in many cases.

(a)Conversion Based on Berry(1968), Berry and Reinhardt(1974) [24]

The method for the conversion based on Berry (1968), Berry and Reinhardt (1974) is shown as following.

$$CN_{cr} = \begin{cases} \dfrac{0.104gE_{cc}}{\mu\left(N_c\rho_w\right)^{\frac{1}{3}}}\left(\bar{\rho}^4 q_c^7\right)^{\frac{1}{3}}, & q_c \geq q_{cm} \\ 0, & q_c < q_{cm} \end{cases} \tag{4.130}$$

where the collection efficiency between cloud water is $E_{cc} = 0.55$. The critical value of mixing ratio of cloud water to convert from cloud water to rain water (q_{cm}) is shown as following.

$$q_{cm} = \frac{\rho_w}{6\bar{\rho}}\pi D_{cm}^3 N_c \tag{4.131}$$

where the critical mean value of diameter of cloud is $D_{cm} = 20$ [μm] . We adopt a typical value ($N_c = 10^8$ [m^{-3}]) of the number concentration of cloud (N_c). However, it is better for this to be calculated by time integration as a forecast variable of a time development equation essentially.

[23]村上 (1999), Lin et al. (1983), Ferrier (1994), Ikawa and Saito (1991)

[24]村上 (1991)

(b)Kessler（1969）'s Definition of Coalescence Growth [25]

Kessler's parametarization is the most classic parameterization, and is also used in many models now.

$$CN_{cr} = a\left(q_c - q_{cm}\right) H\left(q_c - q_{cm}\right) \tag{4.132}$$

where H is set function. Usually a is selected as 10^{-3} $[s^{-1}]$,q_{cm} is selected as 10^{-3}[kg kg^{-1}]. In Cotton and Anthes (1989), q_{cm} and a are given by the function of q_c as following.

$$q_{cm} = \frac{4\pi\rho_w N_c D_{cm}^3}{3\bar{\rho}} = 4\times10^{-12}N_c, \qquad D_{cm} = 10^{-5}\ [m] \tag{4.133}$$

$$a = \pi E_{cc} U_{dc} N_c D_c^2 = 1.3\times q_c^{\frac{4}{3}} N_c^{-\frac{1}{3}}\left(\frac{\rho_0}{\bar{\rho}}\right) \tag{4.134}$$

(c)Lin et al.(1983)'s way

Lin et al. (1983) improved Berry (1968)'s method as following.

$$CN_{cr} = \bar{\rho}\left(q_c - q_{cm}\right)^2\left[1.2\times10^{-4} + 1.569\times10^{-12\frac{N_c}{\sigma^2(q_c-q_{cm})}}\right] \tag{4.135}$$

where σ^2 indicates dispersion of the number concentration of cloud (= 0.15). q_{cm} is selected as 2×10^{-3}[kg kg^{-1}].

Meanings of signs used in (a)∼(c) are shown as following.

g	Acceleration of gravity	9.8	m s^{-2}
N_c	Number concentration of cloud particle	1×10^8	m^{-3}
μ	Viscosity Coefficient of air		kg m^{-1} s^{-1}
$\bar{\rho}$	Air density in the basic state		kg m^{-3}
ρ_0	Air density at the surface in the basic state		kg m^{-3}
ρ_w	Density of water	1×10^3	kg m^{-3}

(2)Conversion from Cloud Ice to Snow： CN_{is}

The conversion from cloud ice to snow is formulized with an assumption of which the conversion occurs by two processes of sublimation growth of cloud ice and condensation[26].

By sublimation condensation growth, it takes Δt_{is1} for ice crystal with $\bar{R}_i$ radius to grow into snow with R_{s0} radius. Δt_{is1} is

$$\Delta t_{is1} = \frac{R_{s0}^2 - \bar{R}_i^2}{2a_1}\rho_i \tag{4.136}$$

where a_1, as well as (4.100) and (4.97), is expressed as following.

$$a_1 = (S_i - 1)\left(\frac{\mathcal{L}_s^2}{\kappa R_v T^2} + \frac{1}{\bar{\rho} q_{vsi}\mathcal{D}_v}\right)^{-1} \tag{4.137}$$

[25] Ikawa and Saito (1991)
[26] 村上 (1999), Murakami (1990), Ikawa and Saito (1991)

Variation of mixing ratio from ice cloud to snow per unit time by sublimation condensation growth (conversion rate) CN_{is}^{dep} is showed as following.

$$CN_{is}^{dep} = \frac{q_i}{\Delta t_{is1}} \tag{4.138}$$

By aggregation growth, it takes Δt_{is1} for cloud ice with $\bar{R}_i$ radius to grow into snow with R_{s0} radius. We suppose that the Δt_{is1} is the same time as it takes for the number concentration of cloud ice to decrease from N_i to $N_i \left(R_i / R_{s0}\right)^3$ on condition $\rho_i = const.$. Therefore, the Δt_{is1} is expressed as following.

$$\Delta t_{is2} = \frac{2}{c_1} \log \left(\frac{R_{s0}}{\bar{R}_i} \right)^3 \tag{4.139}$$

where c_1 is given by the equation (4.122). Variation of mixing ratio from ice cloud to snow per unit time by aggregation growth (conversion rate) is showed as following.

$$CN_{is}^{agg} = \frac{q_i}{\Delta t_{is2}} \tag{4.140}$$

Finaly, the conversion rate from cloud ice to snow (CN_{is}) is given as following.

$$CN_{is} = CN_{is}^{dep} + CN_{is}^{agg} \tag{4.141}$$

Meaning of signs used here are shown as following.

$\mathcal{D}_v$	Diffusion coefficient of water vapor		m^2 s^{-1}
$\mathcal{L}_s$	Latent heat of sublimation of water		J kg^{-1}
q_{vsi}	Saturation mixing ratio over ice		kg kg^{-1}
R_v	Gas constant of water vapor	461.0	J K^{-1} kg^{-1}
$S_i - 1$	Supersaturation of air to ice		
T	Temperature		K
κ	Conduction efficiency of heat of air	2.4×10^{-2}	J m^{-1} s^{-1} K^{-1}
$\bar{\rho}$	Air density in the basic state		kg m^{-3}
ρ_i	Density of cloud ice	5.0×10^2	kg m^{-3}

(3)Conversion from Snow to Graupel: CN_{sg}

We suppose that the conversion from snow to graupel takes place when CL_{cs} (growth rate by caputuring cloud droplets), which is given in (4.109), exceeds VD_{vs} (growth rate by sublimation condensation), which is given in (4.100)[27]。When the conversion takes place, the rate of change of mixing ratio per unit time by converting from snow to graupel is shown as following.

When $CL_{cs} > VD_{vs}$,

$$CN_{sg} = CL_{cs}\frac{\rho_s}{\rho_g - \rho_s}. \tag{4.142}$$

The variation of the number concentration is shown here,

$$CN_{sg}^N = \frac{3\alpha_{us}E_{cs}q_c\left(\rho_0\bar{\rho}\right)^{\frac{1}{2}}\Gamma\left(\beta_{us}\right)}{2\bar{\rho}\left(\rho_g - \rho_s\right)}\lambda_s^{1-\beta_{us}}N_s. \tag{4.143}$$

Meanings of signs used here are shown as following.

E_{cs}	Collection efficiency of snow in the coalescence with cloud ice	$Stk^2 \Big/ (Stk + 0.5)^2$	
α_{us}	The coefficient which appears in the equation of fall velocity and a diameter of snow.	17	m$^{1-\beta_{us}}$ s^{-1}
β_{us}	The coefficient which appears in the equation of fall velocity and a diameter of snow	0.5	
λ_s	Inclination in the reverse exponential function expressing particle size distribution of snow		m^{-1}
$\bar{\rho}$	Air density in the basic state		kg m^{-3}
ρ_0	Air density at the surface in the basic state		kg m^{-3}
ρ_g	Density of graupel	3.0×10^2	kg m^{-3}
ρ_s	Density of snow	8.4×10^1	kg m^{-3}

Melting of Solid Particle: ML

(1)Melting of Cloud Ice: ML_{ic}

[27]村上 (1999), Murakami (1990)

We suppose that cloud ice melts and is converted into cloud water in an instant when $T > T_0$, because the radius of cloud ice (ice crystal) is very small.

When $T_c > T_0$,

$$ML_{ic} = \frac{q_i}{2\Delta t}. \tag{4.144}$$

(2)Melting of Snow and Graupel: ML_{sr}, ML_{gr}

With regard to snow and graupel, ML_{xr} $(x = s, g)$, which indicates the conversion rate from solid particles to rain, is controled by heat budget[28], that is,

When $T > T_0$,

$$ML_{xr} = \frac{2\pi}{\bar{\rho}\mathcal{L}_f}\left[\kappa T_c + \mathcal{L}_v \mathcal{D}_v \bar{\rho}\left(q_v - q_{vs}\left(T_0\right)\right)\right] VENT_x + \frac{C_w T_c}{\mathcal{L}_f}\left(CL_{cx} + CL_{rx}\right). \tag{4.145}$$

If $ML_{xr} > 0$, snow or graupel melts and is converted to rain, however if $ML_{xr} < 0$, even if $T > T_0$, melting does not take place, that is, $ML_{xr} = 0$. $VENT_g$ is ventiration efficiency(4.104).

Meanings of signs used in (1),(2) are shown as following.

CL_{cx}	Growth rate of a particle in x catogory by collision with cloud water		s^{-1}
CL_{rx}	Growth rate of a particle in x catogory by collision with rain water		s^{-1}
C_w	Specific heat at constant pressure of water	4.17×10^3	$J\ K^{-1}\ kg^{-1}$
$\mathcal{D}_v$	Diffusion coefficient of water vapor		$m^2\ s^{-1}$
$\mathcal{L}_f$	Latent heat of fusion of water		$J\ kg^{-1}$
$\mathcal{L}_v$	Latent heat of evaporation of water		$J\ kg^{-1}$
$q_{vs}\left(T_0\right)$	Saturated mixing ratio at melting point of water		$kg\ kg^{-1}$
T	Temperature		K
T_0	Melting point of ice	273.16	K
T_c	Temperature in Celsius scale		°C
κ	Efficiency of conduction of heat of air	2.4×10^{-2}	$J\ m^{-1}\ s^{-1}\ K^{-1}$
$\bar{\rho}$	Air density in the basic state		$kg\ m^{-3}$

[28]Ikawa and Saito (1991)

Freezing of Rain: FR

We suppose that freezing of rain takes place in an instant. All frozen rain is categorized into graupel because we do not consider hail. If there is a category of hail, frozen rain is categorized into the category of hail.

The rate of variation of mixing ratio by freezing rain (FR_{rg}) is based on Bigg (1953)'s empirical equation[29], and is given as following.

$$FR_{rg} = 20\pi^2 B' n_{r0} \frac{\rho_w}{\bar{\rho}} \left[\exp\left(A' T_s\right) - 1\right] \lambda_r^{-7} \tag{4.146}$$

The rate of variation of the number concentration is given as following.

$$FR_{rg}^N = \frac{\pi}{6\bar{\rho}} B' n_{r0} \left[\exp\left(A' T_s\right) - 1\right] \lambda_r^{-4} \tag{4.147}$$

Meanings of signs used here are shown as following.

A'	A coefficient in Bigg's empirical equation	0.66	K^{-1}
B'	A coefficient in Bigg's empirical equation	100.0	$\mathrm{m}^{-3}\ \mathrm{s}^{-1}$
n_{r0}	Density of rain in y section	8.0×10^6	m^{-4}
T	Temperature		K
T_0	Melting point of ice	273.16	K
T_s	Supercooled temperature $(T_0 - T)$		K
λ_r	Inclination in the reverse exponential function expressing particle size distribution of rain		m^{-1}
$\bar{\rho}$	Air density in the basic state		$\mathrm{kg\ m}^{-3}$
ρ_w	Density of water	1×10^3	$\mathrm{kg\ m}^{-3}$

Shedding of Water from Snow and Graupel: SH

In our model, we do not consider that snow and graupel partly contain liquid water in them. Therefore, melted water sheds from snow or ice, and becomes rain water. The process is expressed as following.

When $T > T_0$,

$$SH_{sr} = CL_{cs} + CL_{rs} \tag{4.148}$$

$$SH_{gr} = CL_{cg} + CL_{rg} \tag{4.149}$$

If graupel grows by the wet growth process (even when $T \leq T_0$),

$$SH_{gr} = CL_{cg} + CL_{rg} + CL'_{ig} + CL'_{sg} - PG_{wet}. \tag{4.150}$$

[29] Lin et al. (1983), 村上 (1999)

Meanings of signs used here are shown as following.

CL_{cg}	Growth rate of graupel by collision with cloud water		s^{-1}
CL_{cs}	Growth rate of snow by collision with cloud water		s^{-1}
CL_{rg}	Growth rate of graupel by collision with rain water		s^{-1}
CL_{rs}	Growth rate of snow by collision with rain water		s^{-1}
$CL_{ig}^{'}$	Growth rate of graupel by collision with cloud water in wet growth process		s^{-1}
$CL_{sg}^{'}$	Growth rate of graupel by collision with snow in wet growth process		s^{-1}
PG_{wet}	Growth rate of graupel in wet growth process		s^{-1}
T	Temperature		K
T_0	Melting point of ice	273.16	K

Ferrier (1994) formulizes the wet growth process of snow, graupel and hail (frozen ice) with a time development equation, and takes the wet snow and wet graupel and wet hail into consideration.

Such formulization will be required in the future.

Breakup of Waterdrop

It is known that if it becomes large to some extent (it is 8mm at the diameter of a sphere), rain water becomes unstable and breaks up.

From this reason, the waterdrop more than the size of 8 mm does not fall down in an actual nature. Although the breakup of waterdrop does not change mixing ratio of rain water, it changes number concentration of rain water. However, since the number concentration of rain water is not predicted in this model, the process is not considered.

4.2.5 Moist Saturation Adjustment

The exchange between water vapor and cloud water is expressed by the moist saturation adjustment[30].

We add the mark * to the variable which is calculated by forecast equation. The supersaturated mixing ratio which is defined as

$$\Delta q_c = q_v^* - q_{vsw}^*. \tag{4.151}$$

It $\Delta q_c > 0$, the moist saturation adjustment is applied. The the moist saturation adjustment caluculates $\theta,\ q_v,\ q_c$ with following way.

$$\theta^{t+\Delta t} = \theta^* + \gamma\left(q_v^* - q_{vsw}^*\right) \bigg/ \left[1 + \gamma q_{vsw}^* \frac{4093\Pi}{\left(\Pi\theta^* - 36\right)^2}\right] \tag{4.152}$$

$$q_v^{t+\Delta t} = q_v^* + \left(\theta^* - \theta^{t+\Delta t}\right) / \gamma \tag{4.153}$$

$$q_c^{t+\Delta t} = q_v^* + q_c^* - q_v^{t+\Delta t} \tag{4.154}$$

First, provisional $\theta,\ q_v,\ q_c$ are calculated with these equations. If $q_c^{t+\Delta t} > 0$, the calculation with these equations (4.152)∼(4.154) is repeatedly done until when the value of $\theta,\ q_v,\ q_c$ converges. Therefore, final $\theta,\ q_v,\ q_c$ is decided by the moist saturation adjustment. Here, we defined γ as $\gamma \equiv \mathcal{L}_v / \left(C_p\Pi\right)$.

If $\Delta q_c \leq 0$, we apply following equations.

[30]Soong and Ogura (1973)

$$\theta^{t+\Delta t} = \theta^* - \gamma q_c^* \tag{4.155}$$
$$q_v^{t+\Delta t} = q_v^* + q_c^* \tag{4.156}$$
$$q_c^{t+\Delta t} = 0 \tag{4.157}$$

Meanings of signs used here are shown as following.

C_p	Specific heat at constant pressure of dry air	1004	J K kg^{-1}
$\mathcal{L}_v$	Latent heat of evaporation of water		J kg^{-1}
q_{vsw}	Saturation mixing ratio over water		kg kg^{-1}
Π	Exner function		

4.2.6 Formulization of Variational Terms on Mixing Ratio and Number Concentration with Precipitation

The rate of variation of mixing ratio by fall of particles of cloud and precipitation are shown as following.

$$\text{Fall.}q_x = \frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}\bar{U}_{xq}q_x}{\partial z} \tag{4.158}$$

The mass-weighted terminal fall velocity of particle in category x ($\bar{U}_{xq}$) is given by the equation (4.68) in the section 4.2.3.

The rate of variation of the number concentration is shown as following.

$$\text{Fall.}\frac{N_x}{\bar{\rho}} = \frac{1}{\bar{\rho}}\frac{\partial N_x\bar{U}_{xN}}{\partial z} \tag{4.159}$$

$\bar{U}_{xN}$ indicates mean fall velocity of particle in category x. Similarly, it is given by the equation (4.67) in the section 4.2.3.

When the grid interval in the vertical direction is small, Δt must be enough small so that CFL condition is filled, and so that a falling precipitation particle should not pass next grid for 1 time step.

Since the differentiation in the vertical direction is the differentiation in real space, the differentiation in z^* coordinate （ζ coordinate） must be multiplied by a metric, as shown in equation (2.58).

4.2.7 The other constants

Here, we show variables and constants, which are used in equations but are not explained.

Water Saturation Mixing Ratio and Ice Saturation Mixing Ratio : q_{vsw}, q_{vsi}[31]

$$q_{vsw} = \epsilon \frac{610.78}{p} \exp\left(17.269 \frac{T - T_0}{T - 35.86}\right) \tag{4.160}$$

$$q_{vsi} = \epsilon \frac{610.78}{p} \exp\left(21.875 \frac{T - T_0}{T - 7.86}\right) \tag{4.161}$$

Latent Heat with Evaporation, Sublimation and Melting of Water : $\mathcal{L}_v, \mathcal{L}_s, \mathcal{L}_f$ [J kg^{-1}]

$$\mathcal{L}_v = 2.50078 \times 10^6 \left(\frac{T_0}{T}\right)^{\left(0.167 + 3.67 \times 10^{-4} T\right)} \tag{4.162}$$

$$\mathcal{L}_s = 2.834 \times 10^6 + 100\,(T - T_0) \tag{4.163}$$

$$\mathcal{L}_f = 3.34 \times 10^5 + 2500\,(T - T_0) \tag{4.164}$$

Coefficient of Kinematic Viscosity of Air : ν [m^2 s^{-1}]

$$\nu = 1.328 \times 10^{-5} \frac{p_0}{p} \left(\frac{T}{T_0}\right)^{1.754} \tag{4.165}$$

Coefficient of Viscosity of Air: μ [kg m^{-1} s^{-1}]

$$\mu = \rho\nu \tag{4.166}$$

Coefficient of Water Vapor Diffusion: $\mathcal{D}_v$ [m^2 s^{-1}]

$$\mathcal{D}_v = 2.23 \times 10^{-5} \frac{p_0}{p} \left(\frac{T}{T_0}\right)^{1.81} \tag{4.167}$$

Meanings of signs used here are shown as following.

p	Pressure		Pa
p_0	Standard pressure	101325	Pa
T	Temperature		K
T_0	Melting point of ice	273.16	K
ϵ	Ratio of molecular weight of water vapor to molecular weight of dry air	0.622	
ρ	Air density		kg m^{-3}

[31] Orville and Kopp (1977), Murray (1966)