Chapter 2

Formulation of the system of basic equations

The governing equations of ***CReSS*** is expressed by the equation of motion which is Navier-Stokes equation considering the rotation of the earth, the equation of thermodynamics, the continuity equation with compression, the equation of water vapor mixing ratio, the equation of cloud mixing ratio and precipitation particle and the equation of number density of cloud and precipitation particle. The model is constituted by the formulation of various physical processes and the formulation of boundary value besides their equations. In this chapter, we collect the formulation of basic equations in the constitution.

CReSS can adopt the effect of terrain by taking the grid along terrain. In ***CReSS***, this grid is transformed to the square grid for the calculation. To understand the model easily, first we describe the case of grid in terrain-excluding coordinates. Secondly, we describe the transformation of the grid in terrain-following coordinates and the equations in the case.

2.1 The basic equations system (in terrain-excluding coordinates)

The independent variables of the model are the space coordinates, x, y, z and time, t. In the equations of semi-compression adopted in ***CReSS***, the dependent variables which defined as a function of the independent variables are horizontal velocity, u, v, vertical velocity, w, deviation from the basic status of potential temperature, θ', deviation from the basic status of pressure, p', mixing ratio of water vapor, q_v, mixing ratio of water contents, q_x, and number density of water contents N_x. Here, q_x, N_x are for water contents except water vapor and they are determined how the processes of cloud and precipitation are expressed. Corresponding to this, the number of time-developing equations are varied. Here, dependent variables of potential temperature, pressure and density considering water contents and water vapor are fulfilled by hydrostatic equilibrium,

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g \tag{2.1}$$

The relation between the basic states and the deviation from them are given as,

$$\theta = \bar{\theta} + \theta' \tag{2.2}$$

$$p = \bar{p} + p' \tag{2.3}$$

$$\rho = \bar{\rho} + \rho' \tag{2.4}$$

The density is given by the equation of state diagnostically,

$$\rho = \frac{p}{R_d T}\left(1 - \frac{q_v}{\epsilon + q_v}\right)\left(1 + q_v + \sum q_x\right) \tag{2.5}$$

Here, g is the gravity acceleration, ϵ is the ratio of molecule number between water vapor and dry air and R_d is gas constant of dry air.

All of the dependent variables except density are described in time-developing equations. In the case of the terrain-excluding coordinate, the time-developing equations of the dependent variables are given as follows. In the actual model, these equations are coordinated beside the basic equations which contain terrain in Section 2.2. The terrain-excluding coordinate is used to understand the basic equations easily which result in as follows,

Equation of motion

$$\frac{\partial \bar{\rho} u}{\partial t} = -\bar{\rho}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \frac{\partial p'}{\partial x} + \bar{\rho}\left(f_s v - f_c w\right) + \mathrm{Turb}.u \tag{2.6}$$

$$\frac{\partial \bar{\rho} v}{\partial t} = -\bar{\rho}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) - \frac{\partial p'}{\partial y} - f_s \bar{\rho} u + \mathrm{Turb}.v \tag{2.7}$$

$$\frac{\partial \bar{\rho} w}{\partial t} = -\bar{\rho}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) - \frac{\partial p'}{\partial z} - \bar{\rho}\mathrm{Buoy}.w + f_c u + \mathrm{Turb}.w \tag{2.8}$$

Here, f_s, f_c is Coriolis-parameter and Buoy.w is the term of buoyancy.

Equation of pressure

$$\begin{aligned} \frac{\partial p'}{\partial t} = &-\left(u\frac{\partial p'}{\partial x} + v\frac{\partial p'}{\partial y} + w\frac{\partial p'}{\partial z}\right) + \bar{\rho} g w \\ &-\bar{\rho} c_s^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \bar{\rho} c_s^2\left(\frac{1}{\theta}\frac{d\theta}{dt} - \frac{1}{Q}\frac{dQ}{dt}\right) \end{aligned} \tag{2.9}$$

Here, c_s is the speed of sound in atmosphere and $Q = 1 + 0.61q_v + \sum q_x$.

Equation of potential temperature

$$\frac{\partial \bar{\rho}\theta'}{\partial t} = -\bar{\rho}\left(u\frac{\partial \theta'}{\partial x} + v\frac{\partial \theta'}{\partial y} + w\frac{\partial \theta'}{\partial z}\right) - \bar{\rho}w\frac{\partial \bar{\theta}}{\partial z} + \text{Turb.}\theta + \bar{\rho}\text{Src.}\theta \tag{2.10}$$

Equations of water vapor and mixing ratio of wear contents

$$\frac{\partial \bar{\rho}q_v}{\partial t} = -\bar{\rho}\left(u\frac{\partial q_v}{\partial x} + v\frac{\partial q_v}{\partial y} + w\frac{\partial q_v}{\partial z}\right) + \text{Turb.}q_v + \bar{\rho}\text{Src.}q_v \tag{2.11}$$

$$\frac{\partial \bar{\rho}q_x}{\partial t} = -\bar{\rho}\left(u\frac{\partial q_x}{\partial x} + v\frac{\partial q_x}{\partial y} + w\frac{\partial q_x}{\partial z}\right) + \text{Turb.}q_x + \bar{\rho}\text{Src.}q_x + \bar{\rho}\text{Fall.}q_x \tag{2.12}$$

Equation of number density of water contents

$$\frac{\partial N_x}{\partial t} = -\bar{\rho}\left[u\frac{\partial}{\partial x}\left(\frac{N_x}{\bar{\rho}}\right) + v\frac{\partial}{\partial y}\left(\frac{N_x}{\bar{\rho}}\right) + w\frac{\partial}{\partial z}\left(\frac{N_x}{\bar{\rho}}\right)\right] + \text{Turb.}\frac{N_x}{\bar{\rho}} + \bar{\rho}\text{Src.}\frac{N_x}{\bar{\rho}} + \bar{\rho}\text{Fall.}\frac{N_x}{\bar{\rho}} \tag{2.13}$$

The detail of other dependent variables are described in Section 2.2. The term of diffusion in sub-grid scale Turb.ϕ which appears except in equation of pressure is described in Chapter **??**, and the terms of generation and disappearance Src.ϕ and falling Fall.ϕ which appear in equation of potential temperature and water contents are described in Chapter 4.

2.2 The basic equations system (in terrain-following coordinates)

2.2.1 General curvilinear coordinates

CReSSadopts the terrain-following coordinate to comprise terrain effect. The basis of this coordinate system doesn't necessarily become to orthonormal, while the vector in Cartesian coordinate is expressed by orthogonal basis. This is categorized in curvilinear coordinate of liner algebra. Here, the basic background is summarized.

Contravariant components and covariant components

Introducing general linear independent bases $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, which are neither regular nor orthogonal, facultative vector $\mathbf{A}$ is expressed as the linear combination,

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 \tag{2.14}$$

On the other hand, since the inverse bases $\left(\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3\right)$, is defined as

$$\mathbf{f}^i \cdot \mathbf{e}_j = \delta^i_j \tag{2.15}$$

then, the components are,

$$\mathbf{f}^i \cdot \mathbf{A} = A^i \tag{2.16}$$

Here, Kronecker's delta is defined as follows,

$$\delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.17}$$

In the case of orthonormal basis, coordinate components are given by scalar product of basis vector and facultative vector. In the case of general basis, however, the components must be given by scalar product of inverse basis vector and facultative vector. Thus components of coordinates for original basis are named contravariant components and expressed in a superscript. On the other hand, components of coordinates for inverse basis are named covariant components, expressed in a subscript.

The facultative vector $\mathbf{B}$ is expressed by inverse basic vector as

$$\mathbf{B} = B_1\mathbf{f}^1 + B_2\mathbf{f}^2 + B_3\mathbf{f}^3 \tag{2.18}$$

The solution of scalar product of them becomes

$$\mathbf{A} \cdot \mathbf{B} = A^i B_i \tag{2.19}$$

Thus, scalar product is expressed easily by the sum of product of components in the relative coordinate when both contravariant and covariant components are used. In the case of orthonormal basis, as inverse basis becomes same to original basis, there are no difference between contravariant and covariant components.

$$\mathbf{A} \cdot \mathbf{B} = A^i B^i \tag{2.20}$$

Scalar product of facultative vector $\mathbf{A}$ and bases $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ gives covariant components for bases of vector $\mathbf{A}$, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$,

$$A_i = \mathbf{A} \cdot \mathbf{e}_i \tag{2.21}$$

Next, for bases $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, 9 scalar product can be made,

$$\mathrm{G}_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{2.22}$$

This matrix is named metric matrix. Moreover, this is quadratic tensor which is named metric tensor. Using this, the relationship between contravariant and covariant components is expressed as follows.

$$A^i = \mathrm{G}_{ji} A_j \tag{2.23}$$

curvilinear coordinates

As a function of Cartesian coordinate, three function $F^i(x, y, z)$ which defined at a certain region of a space is considered. The differentiation of them for (x, y, z) is possible for any number of times.

$$\xi = F^1(x, y, z) \tag{2.24}$$
$$\eta = F^2(x, y, z) \tag{2.25}$$
$$\zeta = F^3(x, y, z) \tag{2.26}$$

(ξ, η, ζ) correlates to the each point in a region $P(x, y, z)$. When this correlation is 1 to 1, it can be said that curvilinear coordinates is made in a region. Furthermore, the following condition is assumed.

$$\frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{vmatrix} \neq 0 \tag{2.27}$$

For Cartesian coordinate, vector $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is

$$\mathbf{e}_1 = \begin{pmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{pmatrix} \tag{2.28}$$

Here, vector $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is named 'basis' or 'fundamental vector' of curvilinear coordinates (ξ, η, ζ). Using above, facultative vector field $\mathbf{A}$ is expressed as the linear combinatio of them.

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 \tag{2.29}$$

Then, A^i is named 'contravariant components' of curvilinear coordinates (ξ, η, ζ) of vector field $\mathbf{A}$. The covariant component is mentioned in (2.21). Samely, for the bases $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, 9 scalar product can be made.

$$\mathrm{G}_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{2.30}$$

Matrix G_{ij} is named metric matrix of curvilinear coordinates (ξ, η, ζ). The relationship between contravariant and covariant components is given by (2.23).

2.2.2 Coordinate along terrain

Many cloud models adopt terrain-following coordinate system to consider the effect of terrain. Such coordinate system are oriented on the special relationship of curvilinear coordinates.

CReSSadopts the terrain-following coordinate as well as NHM and ARPS.

$$\xi = x \tag{2.31}$$
$$\eta = y \tag{2.32}$$
$$\zeta = \zeta\left(x, y, z\right) \tag{2.33}$$

In this case, velocity vector of Cartesian coordinate can be expressed as well as (2.29) by components of velocity vector (contravariant components) (U, V, W) in the terrain-following coordinate.(Ordinally, (U, V, W) are contravariant components and they should be expressed in the superscript $\left(u^1, u^2, u^3\right)$. However, since it is easy, it is written in this way.)

$$u = U\frac{\partial x}{\partial \xi} + V\frac{\partial x}{\partial \eta} + W\frac{\partial x}{\partial \zeta} \tag{2.34}$$

$$v = U\frac{\partial y}{\partial \xi} + V\frac{\partial y}{\partial \eta} + W\frac{\partial y}{\partial \zeta} \tag{2.35}$$

$$w = U\frac{\partial z}{\partial \xi} + V\frac{\partial z}{\partial \eta} + W\frac{\partial z}{\partial \zeta} \tag{2.36}$$

Since it is assumed that condition (2.27) is realized, the inverse of the velocity vector can be asked by solving (2.34)$\sim$(2.36) about (U, V, W).

$$UG^{\frac{1}{2}} = uJ^{yz}_{\eta\zeta} + vJ^{zx}_{\eta\zeta} + wJ^{xy}_{\eta\zeta} \tag{2.37}$$

$$VG^{\frac{1}{2}} = uJ^{yz}_{\zeta\xi} + vJ^{zx}_{\zeta\xi} + wJ^{xy}_{\zeta\xi} \tag{2.38}$$

$$WG^{\frac{1}{2}} = uJ^{yz}_{\xi\eta} + vJ^{zx}_{\xi\eta} + wJ^{xy}_{\xi\eta} \tag{2.39}$$

Here, J is Jacobian. For example, it is defined as

$$J^{yz}_{\eta\zeta} \equiv \frac{\partial\left(y, z\right)}{\partial\left(\eta, \zeta\right)} = \begin{vmatrix} \dfrac{\partial y}{\partial \eta} & \dfrac{\partial y}{\partial \zeta} \\ \dfrac{\partial z}{\partial \eta} & \dfrac{\partial z}{\partial \zeta} \end{vmatrix} \tag{2.40}$$

$G^{\frac{1}{2}}$ is defined as Jacobian of the coordinate conversion between (ξ, η, ζ) and (x, y, z) and expressed as

$$G^{\frac{1}{2}} \equiv \frac{\partial\left(x, y, z\right)}{\partial\left(\xi, \eta, \zeta\right)} = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \zeta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial y}{\partial \zeta} \\ \dfrac{\partial z}{\partial \xi} & \dfrac{\partial z}{\partial \eta} & \dfrac{\partial z}{\partial \zeta} \end{vmatrix} \tag{2.41}$$

In the case of terrain-following coordinate defined by (2.31)∼(2.33), Jacobians which appear in (2.37)∼(2.39) are as follows.

$$\begin{aligned}
&J^{yz}_{\eta\zeta} = \frac{\partial z}{\partial \zeta}, && J^{zx}_{\eta\zeta} = 0, && J^{xy}_{\eta\zeta} = 0, \\
&J^{yz}_{\zeta\xi} = 0, && J^{zx}_{\zeta\xi} = \frac{\partial z}{\partial \zeta}, && J^{xy}_{\zeta\xi} = 0, \\
&J^{yz}_{\xi\eta} = -\frac{\partial z}{\partial \xi}, && J^{zx}_{\xi\eta} = -\frac{\partial z}{\partial \eta}, && J^{xy}_{\xi\eta} = 1
\end{aligned} \tag{2.42}$$

In the case of three-dimension, Jacobian of the coordinate conversion between (ξ, η, ζ) and (x, y, z) is

$$G^{\frac{1}{2}} = \left| \frac{\partial z}{\partial \zeta} \right| \tag{2.43}$$

Here, variable components of Jacobian's components are defined as

$$J_{31} \equiv J^{yz}_{\xi\eta} = -\frac{\partial z}{\partial \xi} \tag{2.44}$$

$$J_{32} \equiv J^{zx}_{\xi\eta} = -\frac{\partial z}{\partial \eta} \tag{2.45}$$

$$J_{d} \equiv J^{yz}_{\eta\zeta} = J^{zx}_{\zeta\xi} = \frac{\partial z}{\partial \zeta} \tag{2.46}$$

and ζ is defined by using the altitude of surface $z_{sfc}(x, y)$ and the height of the region of model z_{top},

$$\zeta(x, y, z) = \frac{z_{top}[z - z_{sfc}(x, y)]}{z_{top} - z_{sfc}(x, y)} \tag{2.47}$$

or

$$z(\xi, \eta, \zeta) = z_{sfc}(\xi, \eta) + \zeta \left[1 - \frac{z_{sfc}(\xi, \eta)}{z_{top}} \right] \tag{2.48}$$

In this case, Jacobian's various components are

$$J_{31} = -\frac{\partial z}{\partial \xi} = \left(\frac{\zeta}{z_{top}} - 1 \right) \frac{\partial z_{sfc}(\xi, \eta)}{\partial \xi} \tag{2.49}$$

$$J_{32} = -\frac{\partial z}{\partial \eta} = \left(\frac{\zeta}{z_{top}} - 1 \right) \frac{\partial z_{sfc}(\xi, \eta)}{\partial \eta} \tag{2.50}$$

$$J_{d} = \frac{\partial z}{\partial \zeta} = 1 - \frac{z_{sfc}(\xi, \eta)}{z_{top}} \tag{2.51}$$

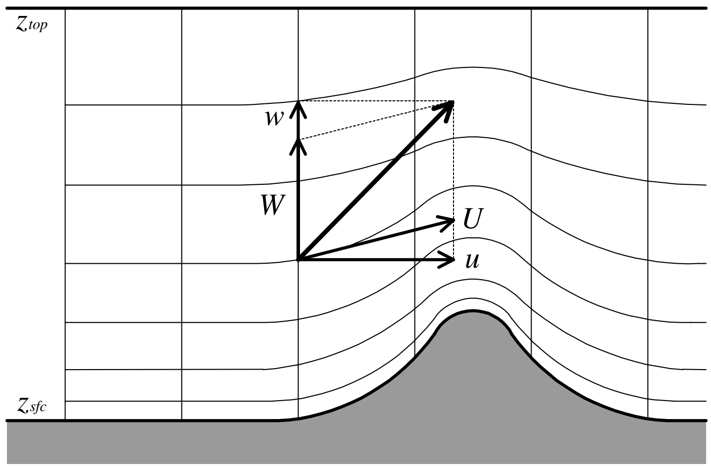

Figure 2.1. Terrain-following coordinates and direction of the vector.

Like in this case, when ζ is a monotonically increasing function about z, it is expressed that

$$G^{\frac{1}{2}} = |J_d| = J_d \tag{2.52}$$

Velocity of the terrain-following coordinate (contravariant velocity) which is given in (2.37)~(2.39) is transformed as follows,

$$U = u \tag{2.53}$$
$$V = v \tag{2.54}$$
$$W = (uJ_{31} + vJ_{32} + w) \Big/ G^{\frac{1}{2}} \tag{2.55}$$

When the transformation from Cartesian coordinate to the coordinate along terrain is performed using above relation, the space differential of a various value ϕ is transformed as follows,

$$\frac{\partial \phi}{\partial x} = \frac{1}{G^{\frac{1}{2}}} \left[\frac{\partial}{\partial \xi} (J_d \phi) + \frac{\partial}{\partial \zeta} (J_{31} \phi) \right] \tag{2.56}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{G^{\frac{1}{2}}} \left[\frac{\partial}{\partial \eta} (J_d \phi) + \frac{\partial}{\partial \zeta} (J_{32} \phi) \right] \tag{2.57}$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{G^{\frac{1}{2}}} \frac{\partial \phi}{\partial \zeta} \tag{2.58}$$

2.2.3 The basic dynamical equations in terrain-following coordinates

In the coordinates of terrain-following, three factors of dependent variables are separated into the values of base state and deviations from the values of base state, as well as in those of terrain-excluding. Those three factors are potential temperature, pressure and density which considered effects of water contents and water vapor. The values of base state are defined so that those are in hydrostatic balance taken account of effects of terrain, which balance is

$$\frac{\partial \bar{\rho}}{\partial \zeta} = -G^{\frac{1}{2}} \bar{\rho} g \tag{2.59}$$

To simplify those representation, we use

$$\rho^* = G^{\frac{1}{2}}\bar{\rho} \tag{2.60}$$

and transform each predicted variable as follows:

$$u^* = \rho^* u \tag{2.61}$$
$$v^* = \rho^* v \tag{2.62}$$
$$w^* = \rho^* w \tag{2.63}$$
$$W^* = \rho^* W \tag{2.64}$$
$$\theta^* = \rho^* \theta' \tag{2.65}$$
$$q_v^* = \rho^* q_v \tag{2.66}$$
$$q_x^* = \rho^* q_x \tag{2.67}$$

Using the above, the basic equations system in terrain-excluding coordinates shown in Section 2.1 of this chapter are transformed in terrain-following coordinates as follows.

Equation of motion

$$\begin{aligned}\frac{\partial u^*}{\partial t} =& -\left(u^*\frac{\partial u}{\partial \xi} + v^*\frac{\partial u}{\partial \eta} + W^*\frac{\partial u}{\partial \zeta}\right) \\ & -\left[\frac{\partial}{\partial \xi}\left\{J_d\left(p' - \alpha Div^*\right)\right\} + \frac{\partial}{\partial \zeta}\left\{J_{31}\left(p' - \alpha Div^*\right)\right\}\right] + \left(f_s v^* - f_c w^*\right) + G^{\frac{1}{2}}\mathrm{Turb}.u\end{aligned} \tag{2.68}$$

$$\begin{aligned}\frac{\partial v^*}{\partial t} =& -\left(u^*\frac{\partial v}{\partial \xi} + v^*\frac{\partial v}{\partial \eta} + W^*\frac{\partial v}{\partial \zeta}\right) \\ & -\left[\frac{\partial}{\partial \eta}\left\{J_d\left(p' - \alpha Div^*\right)\right\} + \frac{\partial}{\partial \zeta}\left\{J_{32}\left(p' - \alpha Div^*\right)\right\}\right] - f_s u^* + G^{\frac{1}{2}}\mathrm{Turb}.v\end{aligned} \tag{2.69}$$

$$\begin{aligned}\frac{\partial w^*}{\partial t} =& -\left(u^*\frac{\partial w}{\partial \xi} + v^*\frac{\partial w}{\partial \eta} + W^*\frac{\partial w}{\partial \zeta}\right) \\ & -\frac{\partial}{\partial \zeta}\left(p' - \alpha Div^*\right) - \rho^*\mathrm{Buoy}.w + f_c u^* + G^{\frac{1}{2}}\mathrm{Turb}.w\end{aligned} \tag{2.70}$$

where the buoyancy term Buoy.w is expressed as

$$\mathrm{Buoy}.w = -g\frac{\rho'}{\bar{\rho}} = g\left(\frac{\theta'}{\bar{\theta}} - \frac{p'}{\bar{\rho}c_s^2} + \frac{q_v'}{\epsilon + \bar{q}_v} - \frac{q_v' + \sum q_x}{1 + \bar{q}_v}\right) \tag{2.71}$$

Here q_v' is not the deviation from the value of base state but the deviation from the initial value, ϵ is the ratio of molecular weight to water vapor and dry air. c_s is the speed of sound in air given by

$$c_s = \sqrt{\gamma R_d \bar{T}}\ , \qquad \gamma \equiv C_p / C_v \tag{2.72}$$

where g is the gravity acceleration、$\bar{T}$ is the temperature of base state and R_d is the gas constant for dry air. C_p, C_v are the specific heat at constant pressure and the specific heat at constant volume for dry air, respectively. f_s, f_c are the Coriolis coefficients:

$$f_s = 2\omega \sin\varphi \tag{2.73}$$
$$f_c = 2\omega \cos\varphi \tag{2.74}$$

where ω is the angular velocity of the earth and φ is the latitude. Furthermore, αDiv^* shown in the pressure term is the divergence damping to suppress soundwaves, which is given by

$$Div^* = \frac{1}{G^{\frac{1}{2}}}\left(\frac{\partial u^*}{\partial \xi} + \frac{\partial v^*}{\partial \eta} + \frac{\partial W^*}{\partial \zeta}\right) \tag{2.75}$$

Equation of pressure

$$\begin{aligned}\frac{\partial G^{\frac{1}{2}} p'}{\partial t} = &-\left(G^{\frac{1}{2}} u \frac{\partial p'}{\partial \xi} + G^{\frac{1}{2}} v \frac{\partial p'}{\partial \eta} + G^{\frac{1}{2}} W \frac{\partial p'}{\partial \zeta}\right) + G^{\frac{1}{2}} \bar{\rho} g w \\ &-\bar{\rho} c_s^2 \left(\frac{\partial G^{\frac{1}{2}} u}{\partial \xi} + \frac{\partial G^{\frac{1}{2}} v}{\partial \eta} + \frac{\partial G^{\frac{1}{2}} W}{\partial \zeta}\right) + G^{\frac{1}{2}} \bar{\rho} c_s^2 \left(\frac{1}{\theta}\frac{d\theta}{dt} - \frac{1}{Q}\frac{dQ}{dt}\right)\end{aligned} \tag{2.76}$$

where $Q = 1 + 0.61 q_v + \sum q_x$ was used.

Equation of potential temperature

$$\frac{\partial \theta^*}{\partial t} = -\left(u^* \frac{\partial \theta'}{\partial \xi} + v^* \frac{\partial \theta'}{\partial \eta} + W^* \frac{\partial \theta'}{\partial \zeta}\right) - \bar{\rho} w \frac{\partial \bar{\theta}}{\partial \zeta} + G^{\frac{1}{2}} \mathrm{Turb.}\theta + \rho^* \mathrm{Src.}\theta \tag{2.77}$$

Equations of mixing ratio of water vapor and water contents

$$\frac{\partial q_v^*}{\partial t} = -\left(u^* \frac{\partial q_v}{\partial \xi} + v^* \frac{\partial q_v}{\partial \eta} + W^* \frac{\partial q_v}{\partial \zeta}\right) + G^{\frac{1}{2}} \mathrm{Turb.}q_v + \rho^* \mathrm{Src.}q_v \tag{2.78}$$

$$\frac{\partial q_x^*}{\partial t} = -\left(u^* \frac{\partial q_x}{\partial \xi} + v^* \frac{\partial q_x}{\partial \eta} + W^* \frac{\partial q_x}{\partial \zeta}\right) + G^{\frac{1}{2}} \mathrm{Turb.}q_x + \rho^* \mathrm{Src.}q_x + \rho^* \mathrm{Fall.}q_x \tag{2.79}$$

Equations of number concentration per unit volume

$$\begin{aligned}\frac{\partial G^{\frac{1}{2}} N_x}{\partial t} = &-\left[u^* \frac{\partial}{\partial \xi}\left(\frac{N_x}{\bar{\rho}}\right) + v^* \frac{\partial}{\partial \eta}\left(\frac{N_x}{\bar{\rho}}\right) + W^* \frac{\partial}{\partial \zeta}\left(\frac{N_x}{\bar{\rho}}\right)\right] \\ &+G^{\frac{1}{2}} \mathrm{Turb.}\frac{N_x}{\bar{\rho}} + \rho^* \mathrm{Src.}\frac{N_x}{\bar{\rho}} + \rho^* \mathrm{Fall.}\frac{N_x}{\bar{\rho}}\end{aligned} \tag{2.80}$$

Similar to terrain-excluding, there are prognostic equations of turbulence kinetic energy E in addition to these equations, which are discussed in Chapter **??** Diffusion of Sub-grid scale. So is the diffusion term of sub-grid scale Turb.ϕ shown in those equations. On the other hand, the production or loss term Src.ϕ and the falling term Fall.ϕ, shown in the equations of potential temperature and water contents, are discussed in Chapter **??** Physical Processes of Clouds and Precipitations.

2.3 基本方程式系 — 地図投影

これまでの雲の数値モデルは地球の曲率を十分無視できる程度の領域で用いられることがほとんどであったので、地球の曲率の効果や地図投影によるゆがみの効果を考慮する必要はなかった。しかしながら、近年の並列コンピューターの大規模化・高速化とともに、雲を解像しながらでも、それらを無視できないほど広い領域での計算が可能になってきた。例えば、日本海を含む領域の計算や、台風を十分広い領域でシミュレーションしようとすると、地球の曲率の効果を無視できなくなり、地図投影が必要になる。その効果を取り入れるために、基本方程式系に**地図系数（map factor）**を導入する。

通常、気象学で用いる地図投影図法や緯度経度座標系は、直交曲線座標系の一つと考えることができ、地形を含まない場合にはデカルト座標から直交曲線座標系への変換を行なえばよい。しかしながら、地形を含む場合は水平方向の座標は直交しているが、それらに対して鉛直座標はもはや直交しておらず、一般の曲線座標となる。

そこで、ここでは一般の曲線座標として座標変換のテンソルを計算するのではなく、簡単のために、まず直交曲線座標系における地形を考えない地図投影の方程式系を求めておいてから、それに地形の効果を取り入れる方法をとる。まずはじめに、直交曲線座標系を用いる場合の微分作用素について整理しておく。

2.3.1 直交曲線座標

第 2.2 節にあるように、直交直線座標系 (x, y, z) の関数として空間のある領域で必要なだけ微分可能な 3 つの一価関数が定義されており、式 (2.27) のようにその関数行列式が 0 でない場合は、曲線座標が定義される。その特別な場合として、基底が作る計量テンソル G_{ij} の対角成分のみがゼロでない場合、その基底は直交しており、そのような曲線座標を**直交曲線座標**という。

緯度経度座標や気象学で通常用いる地図投影座標系は、2 次元の直交曲線座標系とみなすことができ、座標変換のテンソル解析を用いなくてもベクトル解析の範囲で定式化ができる。地図投影は 2 次元であるが、ここでは便宜上水平座標に直交する鉛直座標 z を含めて 3 次元の直交曲線座標を考える。ここでは一般的に直交曲線座標を考えるので、次のような座標系を定義する。

$$\xi = \xi(x, y, z) \tag{2.81}$$

$$\eta = \eta(x, y, z) \tag{2.82}$$

$$\zeta = \zeta(x, y, z) \tag{2.83}$$

さて、第 2.2 節では、正規でも直交でもない 1 次独立な基底 $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ を導入したが、ここでは直交曲線座標を考えるので、これらは単位ベクトルでかつお互いに直交していると仮定する。すなわち、

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{2.84}$$

である。ここでは、基底が直交しているので、反変成分と共変成分は同じになり区別する必要はない。このとき、デカルト座標での線元素 $ds^2 = dx^2 + dy^2 + dx^2$ は、

$$ds^2 = (h_1 d\xi)^2 + (h_2 d\eta)^2 + (h_3 d\zeta)^2 \tag{2.85}$$

と与えられる。ここで、h_i は**メトリック係数（第 1 基本量）**とよばれ、

$$h_1 = \left[\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 + \left(\frac{\partial z}{\partial \xi} \right)^2 \right]^{\frac{1}{2}} \tag{2.86}$$

$$h_2 = \left[\left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 + \left(\frac{\partial z}{\partial \eta} \right)^2 \right]^{\frac{1}{2}} \tag{2.87}$$

$$h_3 = \left[\left(\frac{\partial x}{\partial \zeta} \right)^2 + \left(\frac{\partial y}{\partial \zeta} \right)^2 + \left(\frac{\partial z}{\partial \zeta} \right)^2 \right]^{\frac{1}{2}} \tag{2.88}$$

と与えられるので、ξ 曲線、η 曲線、ζ 曲線の弧長 s_1, s_2, s_3 は、それぞれ、

$$ds_1 = h_1 d\xi \tag{2.89}$$
$$ds_2 = h_2 d\eta \tag{2.90}$$
$$ds_3 = h_3 d\zeta \tag{2.91}$$

と表わされる。

これらを用いて、直交曲線座標 (ξ, η, ζ) における勾配、発散、回転などを与えることができる。まず、任意のスカラー関数を $\phi(\xi, \eta, \zeta)$ とすると、直交曲線座標における勾配は、

$$\nabla \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial \xi} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial \eta} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial \zeta} \mathbf{e}_3 \tag{2.92}$$

と与えられる。ただし、$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ は、直交曲線座標 (ξ, η, ζ) についての正規直交基底（単位接ベクトル）である。

さて、一般に、直交曲線座標の場合、任意のベクトル $\mathbf{A} = A^\xi \mathbf{e}_1 + A^\eta \mathbf{e}_2 + A^\zeta \mathbf{e}_3$ の座標成分 (A^ξ, A^η, A^ζ) だけでなく、その単位接ベクトル $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ も座標 (ξ, η, ζ) の関数であるので、ベクトル量 $\mathbf{A}$ に微分演算子を作用させるときは単位接ベクトルも微分されなければならず、例えば、

$$\frac{\partial A^\xi \mathbf{e}_1}{\partial \eta} = \mathbf{e}_1 \frac{\partial A^\xi}{\partial \eta} + A^\xi \frac{\partial \mathbf{e}_1}{\partial \eta} \tag{2.93}$$

のようになる。また、単位接ベクトルの 9 個の微分 $\dfrac{\partial \mathbf{e}_i}{\partial \xi_i}$ は、それぞれ、

$$\begin{aligned}
&\frac{\partial \mathbf{e}_1}{\partial \xi} = -\frac{1}{h_2}\frac{\partial h_1}{\partial \eta}\mathbf{e}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial \zeta}\mathbf{e}_3, \quad &&\frac{\partial \mathbf{e}_1}{\partial \eta} = \frac{1}{h_1}\frac{\partial h_2}{\partial \xi}\mathbf{e}_2, &&\frac{\partial \mathbf{e}_1}{\partial \zeta} = \frac{1}{h_1}\frac{\partial h_3}{\partial \xi}\mathbf{e}_3, \\
&\frac{\partial \mathbf{e}_2}{\partial \xi} = \frac{1}{h_2}\frac{\partial h_1}{\partial \eta}\mathbf{e}_1, &&\frac{\partial \mathbf{e}_2}{\partial \eta} = -\frac{1}{h_3}\frac{\partial h_2}{\partial \zeta}\mathbf{e}_3 - \frac{1}{h_1}\frac{\partial h_2}{\partial \xi}\mathbf{e}_1, \quad &&\frac{\partial \mathbf{e}_2}{\partial \zeta} = \frac{1}{h_2}\frac{\partial h_3}{\partial \eta}\mathbf{e}_3, \\
&\frac{\partial \mathbf{e}_3}{\partial \xi} = \frac{1}{h_3}\frac{\partial h_1}{\partial \zeta}\mathbf{e}_1, &&\frac{\partial \mathbf{e}_3}{\partial \eta} = \frac{1}{h_3}\frac{\partial h_2}{\partial \zeta}\mathbf{e}_2, &&\frac{\partial \mathbf{e}_3}{\partial \zeta} = -\frac{1}{h_1}\frac{\partial h_3}{\partial \xi}\mathbf{e}_1 - \frac{1}{h_2}\frac{\partial h_3}{\partial \eta}\mathbf{e}_2
\end{aligned}$$

$$\cdots\cdots \tag{2.94}$$

と与えられる。よって、任意のベクトル $\mathbf{A}$ の直交曲線座標における発散は、

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \xi}(h_2 h_3 A^\xi) + \frac{\partial}{\partial \eta}(h_3 h_1 A^\eta) + \frac{\partial}{\partial \zeta}(h_1 h_2 A^\zeta) \right] \tag{2.95}$$

のように、また、回転は次のように与えられる。

$$\nabla \times \mathbf{A} = \begin{vmatrix} \dfrac{\mathbf{e}_1}{h_2 h_3} & \dfrac{\mathbf{e}_2}{h_3 h_1} & \dfrac{\mathbf{e}_3}{h_1 h_2} \\ \dfrac{\partial}{\partial \xi} & \dfrac{\partial}{\partial \eta} & \dfrac{\partial}{\partial \zeta} \\ h_1 A^\xi & h_2 A^\eta & h_3 A^\zeta \end{vmatrix}$$

$$= \frac{\mathbf{e}_1}{h_2 h_3}\left[\frac{\partial\left(h_3 A^\zeta\right)}{\partial \eta} - \frac{\partial\left(h_2 A^\eta\right)}{\partial \zeta}\right] + \frac{\mathbf{e}_2}{h_3 h_1}\left[\frac{\partial\left(h_1 A^\xi\right)}{\partial \zeta} - \frac{\partial\left(h_3 A^\zeta\right)}{\partial \xi}\right] + \frac{\mathbf{e}_3}{h_1 h_2}\left[\frac{\partial\left(h_2 A^\eta\right)}{\partial \xi} - \frac{\partial\left(h_1 A^\xi\right)}{\partial \eta}\right]$$

$$\cdots\cdots \quad (2.96)$$

最後に、任意のスカラー関数 $\phi(\xi,\eta,\zeta)$ のラプラシアンは、

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \xi}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial \eta}\right) + \frac{\partial}{\partial \zeta}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial \zeta}\right)\right] \quad (2.97)$$

と与えられる。

2.3.2 地図投影座標系における基本方程式系 — 地形を含まない場合

地球上の各点は、緯度経度座標や地図投影上の点と 1 対 1 の対応関係を与えることができるので、曲線座標とみなすことができる。さらに、緯度経度座標や気象学で用いる地図投影は、水平方向の座標はお互いに直交しており、鉛直方向を第 3 の座標とすると、地形を含まない基礎方程式系の場合には、これらは 3 次元の直交曲線座標として扱うことができる。ここでは、水平方向の座標を (ξ,η)、鉛直方向の座標を z、それぞれの単位接ベクトルを $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ として、地図投影座標系における基礎方程式を求める。

まず、メトリック係数（拡大因数）を、

$$h_1 = \frac{1}{m} \quad (2.98)$$

$$h_2 = \frac{1}{n} \quad (2.99)$$

$$h_3 = 1 \quad (2.100)$$

としておく。さて、m,n は次の節で地図係数として扱われるが、ここでは、一般性を持たせるために拡大因数としておく。例えば、球面座標であれば、

$$m = \frac{1}{a\cos\phi} \quad (2.101)$$

$$n = \frac{1}{a} \quad (2.102)$$

である。ただし、a は地球の半径、ϕ は緯度である。拡大因数がこれらの形をとると、後に求める拡大因数を含む方程式系は球面の方程式系に帰着できる。

さて、このとき、ある点 P とその近傍の座標を (ξ, η, z) と $(\xi + d\xi, \eta + d\eta, z + dz)$ とすると、各座標軸上の距離 ds_1, ds_2, ds_3 は、

$$ds_1 = \frac{d\xi}{m} \tag{2.103}$$

$$ds_2 = \frac{d\eta}{n} \tag{2.104}$$

$$ds_3 = dz \tag{2.105}$$

また、単位接ベクトルの座標軸方向の微分は、

$$\begin{aligned}
&\frac{\partial \mathbf{e}_1}{\partial \xi} = -n\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\mathbf{e}_2 - \frac{1}{am}\mathbf{e}_3, && \frac{\partial \mathbf{e}_1}{\partial \eta} = m\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right)\mathbf{e}_2, && \frac{\partial \mathbf{e}_1}{\partial \zeta} = 0, \\
&\frac{\partial \mathbf{e}_2}{\partial \xi} = n\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\mathbf{e}_1, && \frac{\partial \mathbf{e}_2}{\partial \eta} = -\frac{1}{an}\mathbf{e}_3 - m\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right)\mathbf{e}_1, && \frac{\partial \mathbf{e}_2}{\partial \zeta} = 0, \\
&\frac{\partial \mathbf{e}_3}{\partial \xi} = \frac{1}{am}\mathbf{e}_1, && \frac{\partial \mathbf{e}_3}{\partial \eta} = \frac{1}{an}\mathbf{e}_2, && \frac{\partial \mathbf{e}_3}{\partial \zeta} = 0
\end{aligned} \tag{2.106}$$

のようになる。以下ではこれらを用いて、運動方程式、熱力学方程式、圧縮系の連続方程式、水蒸気混合比の式、雲・降水粒子の混合比の式、および、雲・降水粒子の数密度の式を書き換える。

まず、運動方程式について考える。地形を含まない場合の運動方程式 (2.6)〜(2.8) をこの直交座標系 (ξ, η, z) で表すことが、ここの目標である。

速度ベクトル $\mathbf{u}$ を (ξ, η, z) の座標で表すと、

$$\mathbf{u} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3 \tag{2.107}$$

であるが、(2.103)〜(2.105) を考慮すると速度の各成分は、次のようになる。

$$u = \frac{ds_1}{ddt} = \frac{1}{m}\frac{d\xi}{dt} \tag{2.108}$$

$$v = \frac{ds_2}{ddt} = \frac{1}{n}\frac{d\eta}{dt} \tag{2.109}$$

$$w = \frac{ds_3}{ddt} = \frac{dz}{dt} \tag{2.110}$$

地図投影における全微分は、

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\xi}{dt}\frac{\partial}{\partial \xi} + \frac{d\eta}{dt}\frac{\partial}{\partial \eta} + \frac{dz}{dt}\frac{\partial}{\partial z} \tag{2.111}$$

であり、式 (2.108)〜(2.110) を考慮すると、

$$\frac{d}{dt} = \frac{\partial}{\partial t} + mu\frac{\partial}{\partial \xi} + nv\frac{\partial}{\partial \eta} + w\frac{\partial}{\partial z} \tag{2.112}$$

となる。よって、曲線座標系における速度 (2.107) の時間微分は、

$$\frac{d\mathbf{u}}{dt} = \frac{du}{dt}\mathbf{e}_1 + \frac{dv}{dt}\mathbf{e}_2 + \frac{dw}{dt}\mathbf{e}_3 + u\frac{d\mathbf{e}_1}{dt} + v\frac{d\mathbf{e}_2}{dt} + w\frac{d\mathbf{e}_3}{dt} \tag{2.113}$$

となり、単位接ベクトルの時間微分を (2.106) および (2.112) を考慮すると、(2.113) の右辺後半の単位接ベクトルの時間微分の項は、

$$\begin{aligned} u\frac{d\mathbf{e}_1}{dt} + v\frac{d\mathbf{e}_2}{dt} + w\frac{d\mathbf{e}_3}{dt} = &-\mathbf{e}_1 mnv\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right] + \mathbf{e}_1\frac{uw}{a} \\ &+\mathbf{e}_2 mnu\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right] + \mathbf{e}_1\frac{vw}{a} - \mathbf{e}_3\frac{u^2+v^2}{a} \end{aligned} \tag{2.114}$$

のようになる。これらの項は座標系が直線でなく、単位ベクトルが場所によって変化することによって現れる項で、**曲率項**あるいは**メトリック項**と呼ばれる。

次に、コリオリ力の項は、$2\mathbf{\Omega}$ の座標系 (ξ, η, z) の各成分を (f_ξ, f_η, f_z) で表すと、

$$\begin{aligned} 2\mathbf{\Omega} \times \mathbf{u} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2\Omega_\xi & 2\Omega_\eta & 2\Omega_z \\ u & v & w \end{vmatrix} \\ &= \mathbf{e}_1(f_\eta w - f_z v) + \mathbf{e}_2(f_z u - f_\xi w) + \mathbf{e}_3(f_\xi v - f_\eta u) \end{aligned} \tag{2.115}$$

また、気圧傾度力は (2.92) より、

$$\nabla p' = m\frac{\partial p'}{\partial \xi}\mathbf{e}_1 + n\frac{\partial p'}{\partial \eta}\mathbf{e}_2 + \frac{\partial p'}{\partial z}\mathbf{e}_3 \tag{2.116}$$

と与えられる。

これらを用いると、運動方程式 (2.6)〜(2.8) は、以下のようになる。

運動方程式

$$\bar{\rho}\frac{\partial u}{\partial t} = -\bar{\rho}\left(mu\frac{\partial u}{\partial \xi} + nv\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial z}\right)$$

$$-m\frac{\partial p'}{\partial \xi} + \bar{\rho}(f_\eta w - f_z v) + \bar{\rho}mnv\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right] - \bar{\rho}\frac{uw}{a} + \mathrm{Turb}.u \quad (2.117)$$

$$\bar{\rho}\frac{\partial v}{\partial t} = -\bar{\rho}\left(mu\frac{\partial v}{\partial \xi} + nv\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial z}\right)$$
$$-n\frac{\partial p'}{\partial \eta} + \bar{\rho}(f_z u - f_\xi w) - \bar{\rho}mnu\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right] - \bar{\rho}\frac{vw}{a} + \mathrm{Turb}.v \quad (2.118)$$

$$\bar{\rho}\frac{\partial w}{\partial t} = -\bar{\rho}\left(mu\frac{\partial w}{\partial \xi} + nv\frac{\partial w}{\partial \eta} + w\frac{\partial w}{\partial z}\right)$$
$$-\frac{\partial p'}{\partial z} - \bar{\rho}\mathrm{Buoy}.w + \bar{\rho}(f_\xi v - f_\eta u) + \bar{\rho}\frac{u^2 + v^2}{a} + \mathrm{Turb}.w \quad (2.119)$$

ここで、$\bar{\rho} = \bar{\rho}(z)$ は基本場の密度で、鉛直座標のみの関数である。また、Buoy.w は浮力項である。

同様にして、気圧偏差の式 (2.9)、温位偏差の式 (2.10)、および、水蒸気や雲物理に関する式 (2.11)〜(2.13) を書きかえると、以下のようになる。

気圧偏差の方程式

$$\frac{\partial p'}{\partial t} = -\left(mu\frac{\partial p'}{\partial \xi} + nv\frac{\partial p'}{\partial \eta} + w\frac{\partial p'}{\partial z}\right) + \bar{\rho}gw$$
$$-\bar{\rho}c_s^2\left[mn\left(\frac{\partial}{\partial \xi}\frac{u}{m} + \frac{\partial}{\partial \eta}\frac{v}{n}\right) + \frac{\partial w}{\partial z}\right] + \bar{\rho}c_s^2\left(\frac{1}{\theta}\frac{d\theta}{dt} - \frac{1}{Q}\frac{dQ}{dt}\right) \quad (2.120)$$

温位偏差の方程式

$$\bar{\rho}\frac{\partial \theta'}{\partial t} = -\bar{\rho}\left(mu\frac{\partial \theta'}{\partial \xi} + nv\frac{\partial \theta'}{\partial \eta} + w\frac{\partial \theta'}{\partial z}\right) - \bar{\rho}w\frac{\partial \bar{\theta}}{\partial z} + \bar{\rho}\mathrm{Src}.\theta + \mathrm{Turb}.\theta \quad (2.121)$$

水蒸気および水物質の混合比の方程式

$$\bar{\rho}\frac{\partial q_v}{\partial t} = -\bar{\rho}\left(mu\frac{\partial q_v}{\partial \xi} + nv\frac{\partial q_v}{\partial \eta} + w\frac{\partial q_v}{\partial z}\right) + \bar{\rho}\mathrm{Src}.q_v + \mathrm{Turb}.q_v \quad (2.122)$$

$$\bar{\rho}\frac{\partial q_x}{\partial t} = -\bar{\rho}\left(mu\frac{\partial q_x}{\partial \xi} + nv\frac{\partial q_x}{\partial \eta} + w\frac{\partial q_x}{\partial z}\right) + \bar{\rho}\mathrm{Fall}.q_x + \bar{\rho}\mathrm{Src}.q_x + \mathrm{Turb}.q_x \quad (2.123)$$

水物質の数密度の方程式

$$
\begin{aligned}
\frac{\partial N_x}{\partial t} &= -\bar{\rho}\left[mu\frac{\partial}{\partial \xi}\left(\frac{N_x}{\bar{\rho}}\right) + nv\frac{\partial}{\partial \eta}\left(\frac{N_x}{\bar{\rho}}\right) + w\frac{\partial}{\partial z}\left(\frac{N_x}{\bar{\rho}}\right)\right] \\
&\quad + \bar{\rho}\mathrm{Src.}\frac{N_x}{\bar{\rho}} + \bar{\rho}\mathrm{Fall.}\frac{N_x}{\bar{\rho}} + \mathrm{Turb.}\frac{N_x}{\bar{\rho}}
\end{aligned} \tag{2.124}
$$

なお、この他に乱流に関わるものとして、乱流運動エネルギーの時間発展方程式と各式に現れる拡散項 Turb.ϕ も変更されなければならないが、これについては第??章「サブグリッドスケールの拡散」で述べる。

2.3.3 地図投影座標系における基本方程式系 — 地形を含む場合

前節までで、地図係数を入れた基本方程式系を求める準備ができたので、本節では、地図係数を含む地形に沿う座標系の基本方程式を求める。先に述べたように地形に沿う座標系を鉛直座標として選ぶと、その座標系はもはや直交座標系にはならず、一般の曲線座標系になる。しかしながら、地図係数は鉛直方向の座標の関数ではないので、ここでは前節で求めた直交曲線座標系に、第 2.2 節の地形の導入と同じ手続きで地形に沿う座標系に直すことができる。

さて、前節までは、m, n は拡大係数として扱ってきたが、ここでは通常気象学で用いられる地図投影の地図係数（map factor）と考える。気象学で用いられる地図投影法は、平射図法（ステレオ図法）、ランベルト正角円錐図法、正角円筒図法（メルカトル図法）などの正角図法である。正角図法とは「角が正しい」または「形が正しい」もので、水平方向の 2 つの座標方向のメトリック係数が地図上のすべての点において等しいものである。すなわち、

$$
\left(h_1 = \frac{1}{m}\right) = \left(h_2 = \frac{1}{n}\right) \tag{2.125}
$$

である。一般には、これらの地図係数は緯度と経度の関数である。

この座標系での全微分は、(2.108)〜(2.110) を考慮すると、

$$
\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial t} + \frac{d\xi}{dt}\frac{\partial}{\partial \xi} + \frac{d\eta}{dt}\frac{\partial}{\partial \eta} + \frac{d\zeta}{dt}\frac{\partial}{\partial z} \\
&= \frac{\partial}{\partial t} + mu\frac{\partial}{\partial \xi} + mv\frac{\partial}{\partial \eta} + W\frac{\partial}{\partial \zeta}
\end{aligned} \tag{2.126}
$$

となり、鉛直速度 W は、

$$
\begin{aligned}
W = \frac{d\zeta}{dt} &= mu\frac{\partial \zeta}{\partial \xi} + mv\frac{\partial \zeta}{\partial \eta} + w\frac{\partial \zeta}{\partial z} \\
&= \left[mu\left(-\frac{\partial z}{\partial \xi}\right) + mv\left(-\frac{\partial z}{\partial \eta}\right) + w\frac{\partial \zeta}{\partial z}\right]\frac{\partial \zeta}{\partial z} \\
&= \frac{1}{G^{\frac{1}{2}}}\left(muJ_{31} + mvJ_{32} + w\right)
\end{aligned} \tag{2.127}
$$

のようになる。また、ある変数 ϕ の空間微分は、

$$
m\frac{\partial \phi}{\partial \xi} \rightarrow m\frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \xi}\left(J_d\phi\right) + \frac{\partial}{\partial \zeta}\left(J_{31}\phi\right)\right] \tag{2.128}
$$

$$m\frac{\partial \phi}{\partial \eta} \rightarrow m\frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \eta}\left(J_d\phi\right) + \frac{\partial}{\partial \zeta}\left(J_{32}\phi\right)\right] \tag{2.129}$$

$$\frac{\partial \phi}{\partial z} \rightarrow \frac{1}{G^{\frac{1}{2}}}\frac{\partial \phi}{\partial \zeta} \tag{2.130}$$

のように変換される。

これらを用いて方程式系を書き換えると、以下のように与えられる。

運動方程式

$$\begin{aligned}\bar{\rho}\frac{\partial u}{\partial t} = &-\bar{\rho}\left(mu\frac{\partial u}{\partial \xi} + mv\frac{\partial u}{\partial \eta} + W\frac{\partial u}{\partial \zeta}\right) - m\frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \xi}\left(J_d p'\right) + \frac{\partial}{\partial \zeta}\left(J_{31}p'\right)\right] \\ &+\bar{\rho}(f_\eta w - f_z v) + \bar{\rho}m^2 v\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{m}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right] - \bar{\rho}\frac{uw}{a} + \mathrm{Turb}.u\end{aligned} \tag{2.131}$$

$$\begin{aligned}\bar{\rho}\frac{\partial v}{\partial t} = &-\bar{\rho}\left(mu\frac{\partial v}{\partial \xi} + mv\frac{\partial v}{\partial \eta} + W\frac{\partial v}{\partial \zeta}\right) - m\frac{1}{G^{\frac{1}{2}}}\left[\frac{\partial}{\partial \eta}\left(J_d p'\right) + \frac{\partial}{\partial \zeta}\left(J_{32}p'\right)\right] \\ &+\bar{\rho}(f_z u - f_\xi w) - \bar{\rho}m^2 u\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{m}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right] - \bar{\rho}\frac{vw}{a} + \mathrm{Turb}.v\end{aligned} \tag{2.132}$$

$$\begin{aligned}\bar{\rho}\frac{\partial w}{\partial t} = &-\bar{\rho}\left(mu\frac{\partial w}{\partial \xi} + mv\frac{\partial w}{\partial \eta} + W\frac{\partial w}{\partial \zeta}\right) \\ &-\frac{1}{G^{\frac{1}{2}}}\frac{\partial p'}{\partial \zeta} - \bar{\rho}\mathrm{Buoy}.w + \bar{\rho}(f_\xi v - f_\eta u) + \bar{\rho}\frac{u^2+v^2}{a} + \mathrm{Turb}.w\end{aligned} \tag{2.133}$$

気圧偏差の方程式

$$\begin{aligned}\frac{\partial p'}{\partial t} = &-\left(mu\frac{\partial p'}{\partial \xi} + mv\frac{\partial p'}{\partial \eta} + W\frac{\partial p'}{\partial \zeta}\right) + \bar{\rho}gw \\ &-\bar{\rho}c_s^2\frac{1}{G^{\frac{1}{2}}}\left[m^2\left(\frac{\partial}{\partial \xi}\frac{G^{\frac{1}{2}}u}{m} + \frac{\partial}{\partial \eta}\frac{G^{\frac{1}{2}}v}{m}\right) + \frac{\partial G^{\frac{1}{2}}w}{\partial \zeta}\right] + \bar{\rho}c_s^2\left(\frac{1}{\theta}\frac{d\theta}{dt} - \frac{1}{Q}\frac{dQ}{dt}\right)\end{aligned} \tag{2.134}$$

温位偏差の方程式

$$\bar{\rho}\frac{\partial \theta'}{\partial t} = -\bar{\rho}\left(mu\frac{\partial \theta'}{\partial \xi} + mv\frac{\partial \theta'}{\partial \eta} + W\frac{\partial \theta'}{\partial \zeta}\right) - \bar{\rho}w\frac{1}{G^{\frac{1}{2}}}\frac{\partial \bar{\theta}}{\partial \zeta} + \bar{\rho}\mathrm{Src}.\theta + \mathrm{Turb}.\theta \tag{2.135}$$

水蒸気および水物質の混合比の方程式

$$\bar{\rho}\frac{\partial q_v}{\partial t} = -\bar{\rho}\left(mu\frac{\partial q_v}{\partial \xi} + mv\frac{\partial q_v}{\partial \eta} + W\frac{\partial q_v}{\partial \zeta}\right) + \bar{\rho}\mathrm{Src}.q_v + \mathrm{Turb}.q_v \tag{2.136}$$

$$\bar{\rho}\frac{\partial q_x}{\partial t} = -\bar{\rho}\left(mu\frac{\partial q_x}{\partial \xi} + mv\frac{\partial q_x}{\partial \eta} + W\frac{\partial q_x}{\partial \zeta}\right) + \bar{\rho}\mathrm{Fall}.q_x + \bar{\rho}\mathrm{Src}.q_x + \mathrm{Turb}.q_x \tag{2.137}$$

水物質の数密度の方程式

$$\begin{aligned}\frac{\partial N_x}{\partial t} =& -\bar{\rho}\left[mu\frac{\partial}{\partial \xi}\left(\frac{N_x}{\bar{\rho}}\right) + mv\frac{\partial}{\partial \eta}\left(\frac{N_x}{\bar{\rho}}\right) + W\frac{\partial}{\partial \zeta}\left(\frac{N_x}{\bar{\rho}}\right)\right] \\ &+\bar{\rho}\mathrm{Src}.\frac{N_x}{\bar{\rho}} + \bar{\rho}\mathrm{Fall}.\frac{N_x}{\bar{\rho}} + \mathrm{Turb}.\frac{N_x}{\bar{\rho}}\end{aligned} \tag{2.138}$$

以上で地形に沿う地図投影座標のの方程式系が求められたが、これらの表記を簡便にするために、アスタリスクの付いた変数 (??)〜(??) を用い、また、

$$m^2\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{m}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right] = u\frac{\partial m}{\partial \eta} - v\frac{\partial m}{\partial \xi} \tag{2.139}$$

であることに注意して、これらを書き換えて形を整えると、以下のようになる。

運動方程式

$$\begin{aligned}\frac{\partial u^*}{\partial t} =& -\left(mu^*\frac{\partial u}{\partial \xi} + mv^*\frac{\partial u}{\partial \eta} + W^*\frac{\partial u}{\partial \zeta}\right) - m\left[\frac{\partial}{\partial \xi}(J_d p') + \frac{\partial}{\partial \zeta}(J_{31}p')\right] \\ &+(f_\eta w^* - f_z v^*) + v^*\left[u\frac{\partial m}{\partial \eta} - v\frac{\partial m}{\partial \xi}\right] - u^*\frac{w}{a} + G^{\frac{1}{2}}\mathrm{Turb}.u\end{aligned} \tag{2.140}$$

$$\begin{aligned}\frac{\partial v^*}{\partial t} =& -\left(mu^*\frac{\partial v}{\partial \xi} + mv^*\frac{\partial v}{\partial \eta} + W^*\frac{\partial v}{\partial \zeta}\right) - m\left[\frac{\partial}{\partial \eta}(J_d p') + \frac{\partial}{\partial \zeta}(J_{32}p')\right] \\ &+(f_z u^* - f_\xi w^*) - u^*\left[u\frac{\partial m}{\partial \eta} - v\frac{\partial m}{\partial \xi}\right] - v^*\frac{w}{a} + G^{\frac{1}{2}}\mathrm{Turb}.v\end{aligned} \tag{2.141}$$

$$\begin{aligned}\frac{\partial w^*}{\partial t} =& -\left(mu^*\frac{\partial w}{\partial \xi} + mv^*\frac{\partial w}{\partial \eta} + W^*\frac{\partial w}{\partial \zeta}\right) \\ &-\frac{\partial p'}{\partial \zeta} - \rho^*\mathrm{Buoy}.w + (f_\xi v^* - f_\eta u^*) + \frac{u^*u + v^*v}{a} + G^{\frac{1}{2}}\mathrm{Turb}.w\end{aligned} \tag{2.142}$$

気圧偏差の方程式

$$\begin{aligned}\frac{\partial G^{\frac{1}{2}}p'}{\partial t} =& -G^{\frac{1}{2}}\left(mu\frac{\partial p'}{\partial \xi} + mv\frac{\partial p'}{\partial \eta} + W\frac{\partial p'}{\partial \zeta}\right) + gw^* \\ &-\bar{\rho}c_s^2\left[m^2\left(\frac{\partial}{\partial \xi}\frac{G^{\frac{1}{2}}u}{m} + \frac{\partial}{\partial \eta}\frac{G^{\frac{1}{2}}v}{m}\right) + \frac{\partial G^{\frac{1}{2}}w}{\partial \zeta}\right] + G^{\frac{1}{2}}\bar{\rho}c_s^2\left(\frac{1}{\theta}\frac{d\theta}{dt} - \frac{1}{Q}\frac{dQ}{dt}\right)\end{aligned} \tag{2.143}$$

温位偏差の方程式

$$\frac{\partial \theta^*}{\partial t} = -\left(mu^*\frac{\partial \theta'}{\partial \xi} + mv^*\frac{\partial \theta'}{\partial \eta} + W^*\frac{\partial \theta'}{\partial \zeta}\right) - \bar{\rho}w\frac{\partial \bar{\theta}}{\partial \zeta} + \rho^*\mathrm{Src}.\theta + G^{\frac{1}{2}}\mathrm{Turb}.\theta \tag{2.144}$$

水蒸気および水物質の混合比の方程式

$$\frac{\partial q_v^*}{\partial t} = -\left(mu^* \frac{\partial q_v}{\partial \xi} + mv^* \frac{\partial q_v}{\partial \eta} + W^* \frac{\partial q_v}{\partial \zeta} \right) + \rho^* \mathrm{Src.}q_v + G^{\frac{1}{2}} \mathrm{Turb.}q_v \tag{2.145}$$

$$\frac{\partial q_x^*}{\partial t} = -\left(mu^* \frac{\partial q_x}{\partial \xi} + mv^* \frac{\partial q_x}{\partial \eta} + W^* \frac{\partial q_x}{\partial \zeta} \right) + \rho^* \mathrm{Fall.}q_x + \rho^* \mathrm{Src.}q_x + G^{\frac{1}{2}} \mathrm{Turb.}q_x \tag{2.146}$$

水物質の数密度の方程式

$$\begin{aligned} \frac{\partial G^{\frac{1}{2}} N_x}{\partial t} = &-\left[mu^* \frac{\partial}{\partial \xi} \left(\frac{N_x}{\bar{\rho}} \right) + mv^* \frac{\partial}{\partial \eta} \left(\frac{N_x}{\bar{\rho}} \right) + W^* \frac{\partial}{\partial \zeta} \left(\frac{N_x}{\bar{\rho}} \right) \right] \\ &+ \rho^* \mathrm{Src.} \frac{N_x}{\bar{\rho}} + \rho^* \mathrm{Fall.} \frac{N_x}{\bar{\rho}} + G^{\frac{1}{2}} \mathrm{Turb.} \frac{N_x}{\bar{\rho}} \end{aligned} \tag{2.147}$$

なお、実際の計算では音波による計算不安定を抑えるため、気圧偏差 p' は人工的に入れた音波の減衰項 αDiv^* を含む、$p' - \alpha Div^*$ で置き換えられる。この項の地図投影座標系における表記は、次のように与えられる。

$$Div^* = \frac{1}{G^{\frac{1}{2}}} \left[m^2 \left(\frac{\partial}{\partial \xi} \frac{u^*}{m} + \frac{\partial}{\partial \eta} \frac{v^*}{m} \right) + \frac{\partial W^*}{\partial \zeta} \right] \tag{2.148}$$

前節ど同様に、乱流運動エネルギーの時間発展方程式と各式に現れる拡散項 $\mathrm{Turb.}\phi$ の変換については、第??章「サブグリッドスケールの拡散」で述べる。

2.3.4 正角投影図法

先にも述べたとおり、一般に領域気象モデルで採用される地図投影は、正角投影法である。それは、拡大係数 m, n の方向性を考えなくてもよくなり、基礎方程式系への導入が容易になるからである。

しかしながら、投影方法により歪の分布は異なるので、地球のどの一部分を切り出しても計算できるようにするためには、種々の投影法を採用しなければならない。***CReSS*** では、計算領域の緯度分布に対して次の 3 種類の投影図法を採用しており、ここでは、それらの詳細について述べる。

- **平射図法（ステレオ図法）：** 高緯度における計算に用いる。
- **ランベルト正角円錐図法　：** 中緯度における計算に用いる。
- **正角円筒図法（メルカトル図法）：** 低緯度における計算に用いる。

平射図法（ステレオ図法）

図より、緯度・経度方向のそれぞれの拡大率は、$p = \dfrac{\pi}{2} - \phi$ を用いて、

$$m_\lambda = \frac{1}{a} \frac{\delta r}{\delta p} \tag{2.149}$$

$$n_\phi = \frac{r\lambda}{a\lambda \sin p} = \frac{r}{a \sin p} \tag{2.150}$$

であるが、この図法は正角投影であり、$m_\lambda = n_\phi$ でなければならないので、

$$\frac{1}{a}\frac{\delta r}{\delta p} = \frac{r}{a \sin p} \tag{2.151}$$

であり、結局、次の微分方程式が得られる。

$$\frac{\delta r}{r} = \frac{\delta p}{\sin p} \tag{2.152}$$

これを解くと、c を積分定数として、

$$r = c\left(\tan\frac{p}{2}\right) \tag{2.153}$$

が得られるが、ある基準緯度 p では歪がないので、式 (2.150) を用いて、

$$n_\phi = \frac{c\tan\frac{p}{2}}{2a\sin\frac{p}{2}\cos\frac{p}{2}} = \frac{c}{2a\cos^2\frac{p}{2}} = 1 \tag{2.154}$$

となる。

ランベルト正角円錐図法

正角円筒図法（メルカトル図法）

緯度 ϕ における緯線方向の拡大率は、

$$n_\phi = \frac{2\pi a}{2\pi a\cos\phi} = \sec\phi \tag{2.155}$$

である。この図法は正角投影であるので、経度方向の拡大率についても、

$$m_\lambda = \sec\phi \tag{2.156}$$

でなければならない。

さて、正角円筒図法では、図で見るとおり、経度方向を x、緯度方向を y に単純に置き換えることができる。今、地図上における基準経度からの経度 λ までの距離を x、赤道からの緯度 ϕ までの距離を y とすれば、

$$x = a\lambda \tag{2.157}$$
$$\delta y = a\sec\phi\ \delta\phi \tag{2.158}$$

書くことができる。y 方向については、微分方程式を解いて、

$$y = a\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right] \tag{2.159}$$

となる。